Using Mutual Information to Characterize Neural PDEs

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Abstract

In this paper we are interested in determining the information bearing properties and abilities of the Hodgkin Huxley equation. For the case of a single axon, we describe precisely how the mutual information between the input and the output of the model can be computed. We extend this to the case of coupled equations and find the intriguing numerical result that mutual information between the input and output variables decreases as ephaptic coupling strength increases. Our results indicate that extracellular effects act as distortion vis-a-vis the information borne by the axons.

1 Introduction and Preliminaries

The mutual information of channels is an important quantifier. Its maximum over "input" distributions, the capacity, quantifies the maximum data rate transmissible over the channel while maintaining a vanishing probability of error. This "dynamical" view is somewhat misleading since the "information" definition of capacity invokes the maximum mutual information between the input and output variables. This is more static than the former view. When we are computing the capacity analytically, the static definition involves finding the relationships between the various random variables and their entropies and mutual informations conditioned or otherwise. However, when we are doing a numerical simulation of the channel and actually transmitting bits which can get corrupted, the dynamical definition makes for more relevance. Regardless of this distinction, even in the analytical setting, capacity considerations often invoke stochastic processes, filtrations and martingales defined on them and therefore explicitly bring in time [1].

The capacity of a neuronal channel has engaged man's attention since at least 1952 [2]. These authors give the formula

$$C = \frac{2}{\triangle T(m+r)} \log_2(m-r) \tag{1}$$

for the capacity of a neuronal link. They also report an average information content of 4.3 bits per impulse.

In [3], Rapoport and Hovrath derive the capacity of a neuron while taking into account its refractory period. They obtain

$$C = \frac{1}{\epsilon(1+a\delta)} \int_{e^{-a\epsilon}}^{1} \frac{\log(1-t)}{t} dt. \tag{2}$$

In [4], Abeles and Lass consider the same problem of the capacity with a refractory period and find the expression

$$C = \frac{1}{\sqrt{2\pi e N}}. \tag{3}$$

In 1970, Sugiyama et. al. [5] derived the form of the inter-spike interval distribution for a spiking neuron, based on which the capacity can be computed [6]. They also presented results on the subthreshold distribution of the neuronal membrane potential. In related work, Sato [7] studies the moments of the interspike interval of a neuron model and in [8] Vidybida looks at this distribution with and without feedback.

S.No.	Symbol	Meaning
1	C	capacitance
2	V	transmembrane voltage
3	(z,t)	(space,time)
4	$G^{ax,my}$	(axonal, myelin) conductance
5	θ	axonal inclination to tract axis
6	$I^{inj,ion}$	(injected, ionic) nodal current
7	$\mathbb{E}$	expectation operator (probabilistic)
8	H	entropy
9	$I(\cdot;\cdot)$	mutual information (m.i.)
10	T	total signal duration

Table 1: Principal symbols and their meanings.

More recently, Ikeda and Manton [9] study the capacity formula for a single spiking neuron channel. They define the capacity per channel use as

$$C_R = \sup_{F \in \mathcal{F}} I_R(F) \tag{4}$$

and study the capacity achieving distribution showing that it is discrete. They also do numerical experiments.

All the above works look at single axons or neurons working in isolation. A natural question that therefore arises is what happens when axons are coupled together and working in unison? It is our aim to address this question by means of analysis and simulations, and we focus only on the mutual information.

This paper is organized as follows. In Section 2 we analyze starting from the case of $N = 1$ axon to a general number N of interacting axons. In Section 3 we present the results of MATLAB simulations or numerical experiments to determine the mutual informations involved. Finally, we conclude with a discussion in Section 4.

Table of Notation

Table 1 succinctly displays all the principal mathematical notation and symbols used in this work.

2 Analysis

The plan for this section is to define the capacity of a single axon in terms of the mutual information between the input and output waveforms. Nodes which bear these waveforms are identified first. The time signal from these nodes is taken as the representative waveform. The signal is Fourier analyzed and the coefficients are treated as random variables since a minor amount of noise is included in the propagation equations at the ionic current update step.

Here we are interested in a tract of N axons [10]. We start with the $N = 1$ case:

$$C\frac{\partial V}{\partial t} = G^{ax}\cos^2\theta\frac{\partial^2 V}{\partial z^2} - G^{my}V + I^{inj,ion} \tag{5}$$

and write its capacity as:

$$Cap = \max I(V(0,[0,T]);V(L,[0,T])). \tag{6}$$

Using $\phi_i(0,T)$ as a C.O.N. set of basis functions [11] we write $V(0,[0,T])$ as[1]

$$\sum_{i=1}^{\infty} v_{0i}\phi_i(0,T). \tag{7}$$

[1]For example, where appropriate, the C.O.N. set of basis functions could be the Fourier basis.

Therefore $V(0,[0,T]) \equiv \{v_{0i}\}_{i=1}^{\infty}$, an equivalence in representation. Similarly, $V(L,(0,T)) \equiv \{v_{Li}\}_{i=1}^{\infty}$. Therefore, if

$$\alpha_i = f_{v_{0i}(v_{0i})}^{max} subj.to P \tag{8}$$

where P is the power constraint, then

$$Cap_i = \alpha_i I(\{v_{0i}\}_{i=1}^{\infty}; \{v_{Li}\}_{i=1}^{\infty}). \tag{9}$$

Now, the condition for complete orthonormality is

$$\int_{-\infty}^{\infty} \quad \phi_i(0,T) \quad \cdot \phi_j(0,T) dt = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{10}$$

and

$$V(0,[0,T]) = \sum_{i=1}^{\infty} v_{0i} \phi_i(0,T). \tag{11}$$

Project both sides of Equation (11) onto the CON set, to get

$$\int_{-\infty}^{\infty} \phi_j(0,T) V(0,[0,T]) = \int_{-\infty}^{\infty} \phi_j(0,T) \sum_{i=1}^{\infty} v_{0i} \phi_i(0,T)$$

Switching the order of the summation and integration, we have

$$\begin{array}{rcc} = & \sum_{i=1}^{\infty} v_{0i} \int_{-\infty}^{\infty} \phi_i \phi_j dt & \\ = & \sum_{i=1}^{\infty} v_{0i} \begin{array}{c} 0, i \neq j \\ 1, i = j \end{array} & \\ = & v_{0j} & \end{array} \tag{12}$$

Therefore,

$$v_{0j} = \int_{-\infty}^{\infty} \phi_j(0,T) \cdot V(0,[0,T]) dt \tag{13}$$

and

$$v_{Lj} = < \phi_j(0,T), V(L,[0,T]) > . \tag{14}$$

2.1 Determining $H(v_{0j})$

Assume v_{0j} belong to a discrete set. This is possible if there are only a few stimulus waveforms. In this scenario, let $v_{0j} \in [v_{0j}^1, v_{0j}^2, \ldots, v_{0j}^k]$. Then,

$$H(v_{0j}) = -\sum_{i=1}^{k} p_i \log p_i \tag{15}$$

where $p_i = Pr(v_{0j} = v_{0j}^i)$. Assuming a uniform prior, we have $p_i = \frac{1}{k}$. Call this, assumption (**). This implies,

$$H(v_{0j}) = -\sum_{i=1}^{k} \frac{1}{k} \log(\frac{1}{k}), \tag{16}$$

where k is the total number of possible stimuli.

2.2 Determining $H(v_{0j}|v_{Lj})$

By definition,

$$H(v_{0j}|v_{Lj}) = -\mathbb{E}_{\mathrm{v_{Lj}}}[\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{Pr(v_{0j}^{i}|v_{Lj})} \cdot \mathrm{log(Pr(v_{0j}^{i}|v_{Lj}))}] \tag{17}$$

2.3 Determining $Pr(v_{0j}^{i}|v_{Lj})$

As per Bayes' formula [12],

$$Pr(v_{0j}^{i}|v_{Lj}) = \frac{Pr(v_{Lj}|v_{0j}^{i}) \cdot Pr(v_{0j}^{i})}{Pr(v_{Lj})}. \tag{18}$$

Using the uniform prior, this yields

$$Pr(v_{0j}^{i}|v_{Lj}) = \frac{Pr(v_{Lj}|v_{0j}^{i})}{k \cdot Pr(v_{Lj})}. \tag{19}$$

Further suppose that v_{Lj} also belong to a discrete set of size F. Then, $Pr(v_{Lj}|v_{0j}^{i})$ which stands for $Pr(v_{Lj} = v_{Lj}^{m}|v_{0j}^{i})$, can be equated with $\beta_{L|0}(m|i)$ which is the transition matrix of the 'channel.'

Next, we use the total probability theorem [13] to write,

$$Pr(v_{Lj} = v_{Lj}^{m}) = \sum_{i=1}^{k} \beta_{L|0}(m|i) \cdot Pr(i) \tag{20}$$

Now,

$$Pr(v_{0j}^{i}|v_{Lj}^{m}) = \frac{Pr(v_{Lj}|v_{0j}^{i})}{k \cdot Pr(v_{Lj}^{m})} = \frac{\beta_{L|0}(m|i)}{\sum_{n=1}^{k} \beta_{L|0}(m|n)} \tag{21}$$

This allows us to determine $H(v_{0j}|v_{Lj})$. Thus, we in essence have $I(v_{0i}; v_{Li})$.

2.4 $N = 2$ case and higher

We first write out the coupled equations in this setting. Following [14] we have

$$C_1 \frac{\partial V_1}{\partial t} = G_1^{ax} \cos^2\theta_1 \left[\frac{\partial^2 V_1}{\partial z^2} - W_{11}K_1 \frac{\partial^2 V_1}{\partial z^2} - W_{12}K_2 \frac{\partial^2 V_2}{\partial z^2}\right] - G_1^{my} V_1 + I_1^{inj} \tag{22}$$

and

$$C_2 \frac{\partial V_2}{\partial t} = G_2^{ax} \cos^2\theta_2 \left[\frac{\partial^2 V_2}{\partial z^2} - W_{21}K_1 \frac{\partial^2 V_1}{\partial z^2} - W_{22}K_2 \frac{\partial^2 V_2}{\partial z^2}\right] - G_2^{my} V_2 + I_2^{inj}. \tag{23}$$

These equations are clearly coupled to one another. They form the basis of (current-mediated) ephaptic coupling in which a return path through the extracellular space causes coupling between the axonal transmembrane potentials. Extending the development in the previous subsections, we can write the capacity as

$$Cap_{N=2} = \max I(\{V_1(0,[0,T]), V_2(0,[0,T])\}; \{V_1(L,[0,T]), V_2(L,[0,T])\}). \tag{24}$$

The mutual information can then be expanded as the difference between the output entropy $H(\{V_1(L,[0,T]), V_2(L,[0,T])\})$ and the conditional output entropy $H(\{V_1(L,[0,T]), V_2(L,[0,T])\} \,|\, \{V_1(0,[0,T]), V_2(0,[0,T])\})$. This is straightforward to extend to the case of higher N.

Algorithm 1 Steps for numerical investigations.

1. Set $N = 2$
2. Set the parameter to be studied, diameter or ratio.
3. Execute the program from [14] with these settings.
4. Save the input node waveform and output node waveform for both axons.
5. Repeat 1 through 4, 100 times.
5. Extract the complex Fourier coefficients of these waveforms.
6. Find the empirical pmfs of the input and the output, as well as the joint pmf.
7. Use the empirical pmfs to compute the mutual information from the empirical entropies.

3 Numerical Simulations

The algorithm to perform the numerical simulations outlined in this section is presented as Algorithm 1.

3.1 Relation Between Mutual Information and Coupling Strength

With the description of the mutual information in hand, we perform numerical simulations to determine lower bounds on the capacity for various configurations. The results are shown in Table 2 and Figure 1. Note that "ratio" is directly proportional to the extracellular resistivity which is related directly to the ephaptic coupling strength. We find a "convex hull" relationship between mutual information and coupling strength.

N	number of runs	rf	ratio	mutual information (bits per $N = 2$ tract use)
2	100	1.27e8	0.05	3.8862
2	100	1.27e8	0.2	3.7949
2	100	1.27e8	0.35	3.7385
2	100	1.27e8	0.5	3.701
2	100	1.27e8	0.8	3.6706
2	100	1.27e8	0.95	3.7013

Table 2: Variation of Mutual Information With Strength of Ephaptic Coupling.

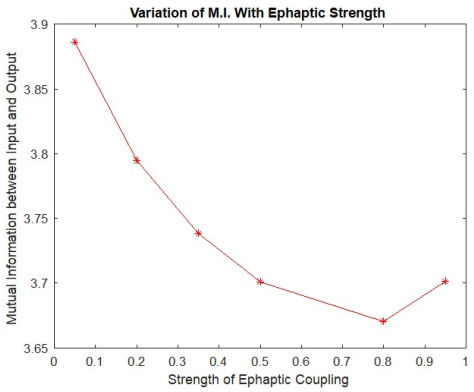

Figure 1: Variation of mutual information with ephaptic coupling strength. The plot resembles the convex hull seen in R-D plots in rate distortion theory. The horizontal axis is the strength of ephaptic coupling and the vertical axis is the mutual information between the input and the output of the tract of two coupled axons.

Furthermore, the mutual information obtained in Table 2 is of the same order of magnitude as reported by MacKay and McCulloch [2] for single neurons, thereby substantiating our results.

3.2 Relation Between Mutual Information and Axon Diameter

Next we fix the coupling strength (ratio = 0.2) and vary the axon diameter between 1e-3 cm and 1.5e-3 cm, determining the mutual information averaged over 100 runs.

N	number of runs	rf	diameter of each axon (cm)	mutual information
2	100	1.27e8	1e-3	3.7949
2	100	1.27e8	1.1e-3	3.7227
2	100	1.27e8	1.2e-3	3.6771
2	100	1.27e8	1.3e-3	3.6473
2	100	1.27e8	1.4e-3	3.6748
2	100	1.27e8	1.5e-3	3.6775

Table 3: Variation of Mutual Information With Strength of Ephaptic Coupling.

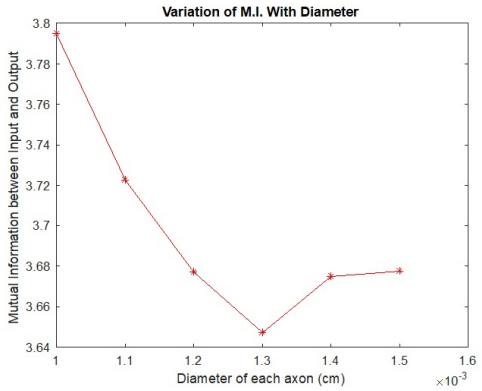

Figure 2: Variation of mutual information with the diameter of each axon. The parameter "ratio" is fixed at 0.2. The horizontal axis is the diameter of the axons and the vertical axis is the mutual information between the input and the output of the tract of two coupled axons.

4 Discussion, Limitations and Conclusion

The result of this paper that there is an inverse relation between ephaptic coupling strength and capacity (more precisely, we only looked at the mutual information in Figure 1) could be due to the fact that "variation" amongst the signals decreases with increased coupling, as synchronized propagation tends to increase. However, it must be acknowledged that towards higher coupling strengths, the relation was no longer inverse. A similar, largely inverse relation, was noted between the mutual information and fiber diameter. It is well known that the larger the diameter, the greater the conduction velocity (see, for example, [15]). However, the mutual information seems to buck this trend, at least in the case of a tract of two axons that are ephaptically coupled (Figure 2).

One of the key limitations in our work is that we did not address the maximization problem which must be addressed in order to find the capacity. This might be addressable using a version of the Blahut-Arimoto algorithm, but it is left for future investigation. Future work will also perform studies of higher N, and more detailed investigation of the relation between geometry and mutual information.

In summary, we utilized the m.i. as a way of characterizing the data transmission capability of a tract of $N = 2$ axons in particular, and any N in general. Using a Fourier decomposition of the input and output signals, we were able to characterize their mutual information via the empirical entropy. Other C.O.N. bases could also be used. Our method could be exploited for other, non-neural PDEs as well. The paper sets the stage for an investigation of the relation between m.i. and geometry (for higher N), as well as

m.i. and tortuosity. Overall, it is a further step in the direction of showing how information theory can be meaningfully applied in neurophysiology.

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