# Towards a Dynamical Systems Approach to Feedback Communication

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*Abstract*—**In this proposal, the authors study the reliability function of a Gaussian noise communication system with Gaussian noise in the feedback link. The noisy feedback error exponent is proposed to be upper bounded by drawing an analogy with random dynamical systems. A sketch of the path forward is outlined, utilizing parallels between dynamical systems theory and information theory.**

***This proposal is dedicated to Prof. Sanjoy K. Mitter (1933-2023).***

*Index Terms*—**Gaussian, feedback, noisy channel, reliability function, Lyapunov exponent.**

## I. Introduction and Problem Setup

In [1], the author determined a lower bound on the reliability function of a Gaussian noise channel with Gaussian noisy feedback. The aim of the present note is to propose a path to an upper bound on the very same reliability function.

Fig. 1 shows the events which occur in a noisy feedback communication system with time increasing from left to right.

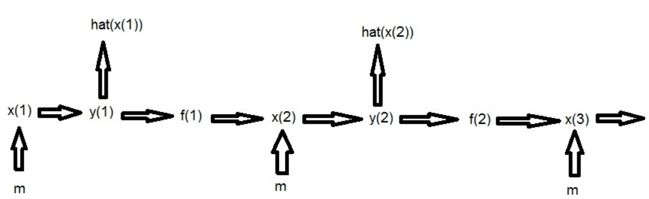


Fig. 1. Time evolution of a random dynamical communication system with noisy feedback

In Fig. 2, which is for the noiseless or perfect feedback case, the $y$ and $f$ stochastic processes have been equated.

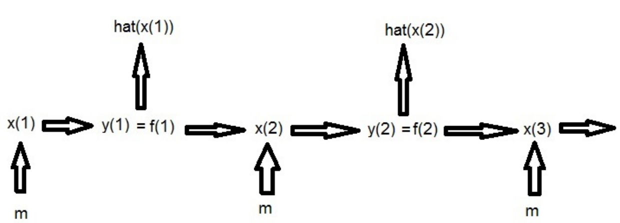


Fig. 2. Time evolution of a random dynamical communication system with noiseless feedback

In the figures, the arrows indicate dependencies between the stochastic processes. In Fig. 3, in the "$g$" sequence, there are two component "sub-sequences":

1) The "odd" subsequence which is the $x$-sequence or input sequence, and
2) The "even" subsequence which is the $y/f$-sequence or output sequence.

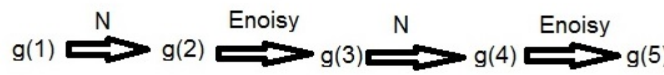


Fig. 3. Time evolution of a random dynamical communication system with noiseless feedback

In another view, if we want to just look at the inputs, we can simply observe the $x$-sequence as in the following figure, Fig. 4.

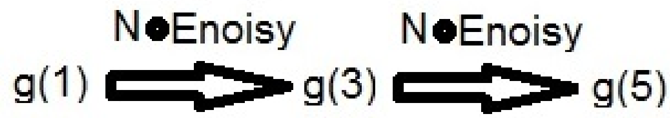


Fig. 4. Time evolution of a random dynamical system based on the inputs to the corresponding communication system with noiseless feedback.

If we relabel $g(1)$ as $h(1), g(3)$ as $h(2)$, and so on, upto $g(2n-1)$ as $h(n)$, then we obtain the next figure, Fig. 5.

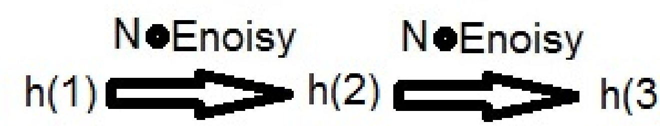


Fig. 5. Time evolution of a random dynamical system based on the inputs to the corresponding communication system with noiseless feedback.

Define $\phi = \mathcal{N} \cdot \mathcal{E}$. Then $\phi$ is a dynamical system.

In an alternate approach, the following figure, Fig. 6, looks at the even subsequence, which is the output.

Next, we let $\alpha(1)$ be $g(2)$, $\alpha(2)$ be $g(4)$, $\alpha(3)$ be $g(6)$ and so on until we let $\alpha(n) = g(2n)$, where $n$ is a positive integer. Further, we let $\phi^{'} = \mathcal{E} \cdot \mathcal{N}$. As a result, we have the mapping from $\alpha(1)$ to $\alpha(2)$, called $\psi^{'}$. Thus, $\alpha(2) = \psi^{'}(\alpha(1))$ and in general,

$$\alpha(n) = \psi^{'}(\alpha(n-1)) \tag{1}$$

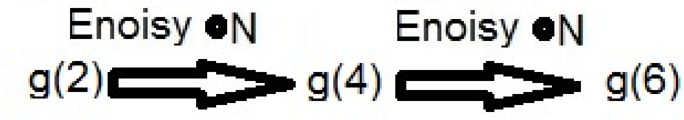


Fig. 6. Time evolution of a random dynamical system based on the inputs to the corresponding communication system with noiseless feedback.

The random dynamical system of 1 will be having statistics which are independent of time because $\mathcal{N}$ is time-independent and also $\mathcal{E}$ has time-independent statistics.

## II. Noisy Feedback Case

If we consider the noisy feedback case, then $f(1) \neq y(1)$ but rather $f(1) = y(1) + noise$. So the encoder can be treated as adding this noise before using $y(1)$ to generate $x(2)$. So it is a noisy encoder $\mathcal{E}_{noisy}$. Thus $\psi^{''} = \mathcal{E}_{noisy} \cdot \mathcal{N}$ and so

$$\beta(n) = \psi^{''}(\beta(n-1)). \tag{2}$$

The random dynamical system of (2) will also have statistics which are independent of time.

## III. Lyapunov Exponent Intuition

In this section we present some intuition on the Lyapunov exponent as it relates to our problem setup. The Lyapunov exponent of our RDS $\psi^{''}$ will be high $\iff$ the system diverges in phase space $\iff$ two close-by initial conditions lead to divergent trajectories of the system.

But, if the two trajectories are divergent, for close-by initial conditions, then resolution of decoding would have increased. That is, you would easily be able to say, given a trajectory, which initial condition it came from, or at least, that it is from a distinct set of possible initial conditions.

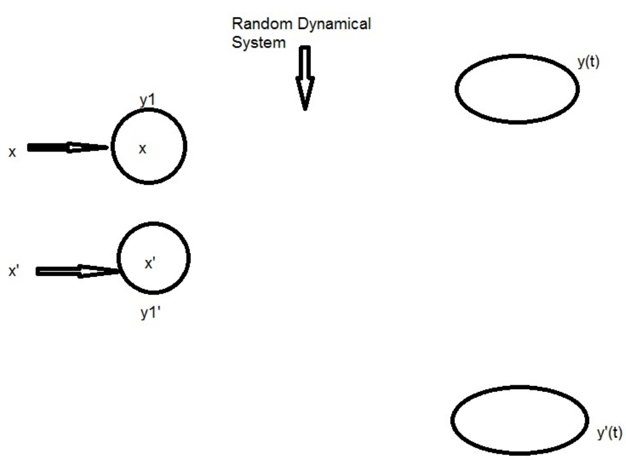


Fig. 7. Divergence of paths starting from nearby initial conditions.

In Figure 7, even if the clouds $y1$ and $y1'$ had some small overlap, a large Lyapunov exponent would lead to a huge gap in a short time between the resultant clouds. Under ML decoding, this would imply a very low probability of error in short time, or a huge error exponent.

## IV. Application of Intuition

Consider a channel with noisy feedback, such as the Discrete Memoryless Channel (DMC), the Gaussian channel or a Vector Gaussian channel, which models an ephaptic tract [2]. As per the development in the preceding sections, we can model these channels as random dynamical systems and find the corresponding Lyapunov exponent (LE). The larger the LE, the larger would be the error exponent.

As a procedure, we can first determine the RDS corresponding to the communication system, then perturb this so that we get a 'closest' chaotic communication system. That latter dynamical system's Lyapunov exponent will be proportional to a tight upper bound on the original random dynamical communication system's error exponent. Here the assumption is that chaotic systems are an altogether different class of dynamical systems.

## V. Finding the Lyapunov Exponent

Our first step is to find the Lyapunov exponent of the random dynamical system shown in Equation 2. For this we will need an application of the multiplicative ergodic theorem [3] and the book [4]. Before we can write the Lyapunov exponent in the most general case, we need the map $\psi^{'}$. We can specialize to the binary case. That is, we consider only antipodal signaling over the AWGN channel on the forward link as well as the feedback link. Thus the noise is Gaussian but at each step we have either a '0' (just a label for '-1') or a '1'.

The most general non-noisy encoder would be

$$\mathcal{E}_t : (m, Y_1^t) \rightarrow X_t \tag{3}$$

Here $m$ is the message, which can also belong to a binary set (two possibilities) and $Y_1^t$ is a length $t$ bit sequence. This thus represents a feedback encoder. We can assume initially that there is only a length 1 feedback. However, for illustration purposes (see Figure 8), let us consider a length 2 feedback at each step. That is, it is a fixed length feedback code instead of a variable length one. For this feedback code, we can write

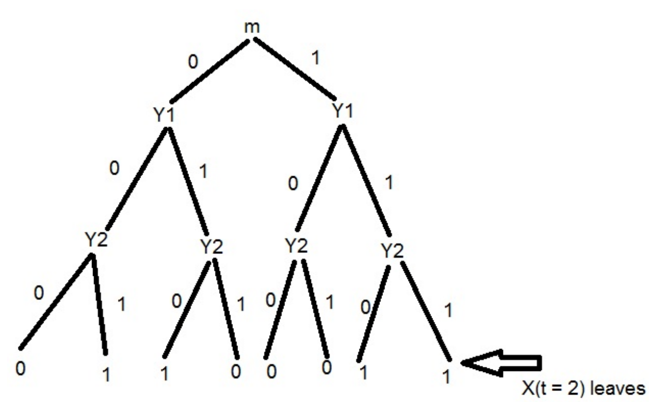


Fig. 8. Illustration of a binary feedback code.

$$
\begin{aligned}
X(t=2) &= m' \cdot y_1^{'} \cdot y_2 + m' \cdot y_1 \cdot y_2^{'} + m \cdot y_1 \cdot y_2^{'} \\
&+ m \cdot y_1 \cdot y_2 \\
&= m'(y_1(xor)y_2) + m \cdot y_1
\end{aligned} \quad (4)
$$

We can write the above succinctly as

$$
X(t) = \sum_{a,b,c} \mathbb{K}_{a,b,c} \cdot m(t-1)^a \cdot y(t-1)^b \cdot y(t-2)^c \quad (5)
$$

where $\{a, b, c\} \in \{prime, non-prime\}$ and $\mathbb{K}_{a,b,c}$ is a coefficient which is either 0 or 1 and indicates if the corresponding product term is present or not.

For equating with a Random Dynamical System, the encoder has to be a fixed mapping applied at every instant. So, our present formulation is applicable perhaps only to fixed length codes as opposed to variable length codes. This is fine to begin with, but this assumption will have to be relaxed to find the most general upper bound. Further, a new theory, broadening that of Random Dynamical Systems as defined in Arnold's book [4], might have to be developed [1].

Next we look at the channel. Let it be an additive noise channel. The input-output mapping is therefore the identity map, perturbed by noise. Thus,

$$
\begin{aligned}
y(t) &= x(t) + noise \\
&= \mathbb{I} \cdot x(t) + noise
\end{aligned} \quad (6)
$$

Thus we can finally write $\psi^{'}$ for this special case as $\psi^{'} = \mathcal{E} \cdot \mathcal{N}$. This gives us

$$
\psi^{'} = \sum_{a,b,c} \mathbb{K}_{a,b,c} \cdot m(t-1)^a \cdot y(t-1)^b \cdot y(t-2)^c \quad (7)
$$

to which noise would be added. This is the Random Dynamical System (RDS) corresponding to the output $y$ of the channel. Our next step would be to consider the binary case where distances are Hamming distances and then to find the Lyapunov exponent which would tell us about the divergence rates of trajectories. We will have to find the perturbed RDS of $\psi^{'}$ with the smallest Lyapunov exponent such that the system is still chaotic in order to upper bound the reliability function.

## VI. Conclusion

In this note, we have presented initial ideas in an approach towards understanding noisy feedback converses to various types of channels, in relation to their error exponent or reliability function. Our approach leads to a merger between the fields of dynamical systems theory and information theory and in future work, we will take this program to its logical conclusion.

## References


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[1] Note however that if a random dynamical system is described in terms of a triple, then within the choice of maps we can include variable length codes [5].