Revisiting the 3D Tract Equation for Generic Axonal Geometry

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Abstract

In this note the authors take steps towards handling generically placed axons in a cylindrical tract.

The Hodgkin Huxley equation describes a single axon. One can rederive the cable equation in a three dimensional space wherein Kirchoff's laws are combined with a scaled partial derivative where the scaling factor contains the inclination angle of the axon with respect to the vertical z-axis, or the bounding cylinder's central axis. For axons that lie in a vertical plane which contains the entire z-axis, this formulation is sufficient.

The most generic axon however, will lie in a vertical plane which may not intersect with the z-axis at all. Its containing plane or c-plane will instead intersect with the xz and yz planes. If we can formulate a modified cable equation for a nominal and a generic axon then we will have an approach to extending the formulation to arbitrarily large tracts of N axons.

The key point where we must focus our attention is on the scaled partial derivative's scaling factor. Instead of taking the z-axis as defining the universal coordinate, we can utilize the xz and yz planes. We can project the generic axon onto the xz plane and then onto the yz plane. From these two planes since they are vertical and contain the z-axis, we can then project onto the z-axis.

So consider a generic axon r making an angle a with the xz plane and an angle b with the yz plane. The projection onto the xz plane, rxz, has length $r * cos(a)$ and the projection onto the yz plane, ryz, has the length $r * cos(b)$. Next we project rxz onto the z-axis where c is the angle between rxz and the z-axis. This yields a segment $rxzz$ of length $rxz * cos(c)$. Similarly the projection of ryz onto the z-axis where d is the angle between ryz and the z-axis, yields a segment $ryzz$ of length $ryz * cos(d)$.

Thus, instead of tracking a single projection for each nominal axon as was done in our 2019 paper, we must track two projections for each generic axon. Effectively, it is as though the number of axons has been doubled to $2 * N$, in the case where all are generic and none are nominal. The two projections $rxzz$ and $ryzz$ are of length $r * cos(a) * cos(c)$ and $r * cos(b) * cos(d)$, respectively. Let all the generic axonal projections via the xz plane be assigned to the first universal coordinate $z1$ and all the generic axonal projections via the yz plane be assigned to the second universal coordinate $z2$. Then, $z1 = r_i * cos(a_i) * cos(c_i)$ where i runs from 1 through N and $z2 = r_j * cos(b_j) * cos(c_j)$ where j runs from 1 through N.

We repeat the derivation of the 2019 paper from this point onwards, first for $z1$ and next for $z2$. In the end, we can form the transmembrane voltage V for an axon using that obtained via the $z1$ route, $V1$ and via the $z2$ route, $V2$. This is a non-trivial exercise in finding the original signal (in a $3D$ space) based on two projections (lying in $2D$ spaces) or partial views. The reconstruction has to be consistent with both the partial views. The solution may even be a family of possibilities, instead of one single consistent reconstruction.