Programming Hilbert Space

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Abstract—In this note, the authors investigate whether dynamic programming can be applied to L'Hopital's rule. We present a novel computational framework for evaluating limits of indeterminate forms using dynamic programming principles. Traditional approaches such as L'Hôpital's rule rely on symbolic differentiation and analytical manipulation, which may be unavailable or computationally intensive for complex functions. Our method reformulates limit evaluation as a multi-stage decision process where each stage represents a strategic approach toward the limit point. The algorithm optimally balances accuracy, computational cost, and numerical stability through recursive functional equations derived from Bellman's principle of optimality. We demonstrate this approach on classical indeterminate forms including $0/0$, ∞/∞, and $0 \cdot \infty$, showing competitive accuracy with reduced symbolic complexity. The framework naturally handles black-box functions, adaptively selects step sizes, and provides confidence bounds on numerical estimates. This work bridges classical optimization theory with numerical analysis, offering practitioners an alternative tool for limit evaluation in computational settings where symbolic methods are impractical.

Index Terms—dynamic programming, limit evaluation, L'Hôpital's rule, numerical analysis, optimization, indeterminate forms

> Hilbert space[1] is gratuitously big—much bigger than the space needed to carry $\log_2(D)$ bits. Yet, because no measurement can distinguish all these states, almost none of this huge amount of information is accessible to observation [1].

I. Introduction

The evaluation of limits [2] forms a cornerstone of mathematical analysis, with applications spanning engineering, physics, economics, and computer science. When direct substitution yields indeterminate forms such as $0/0$ or ∞/∞, classical calculus provides L'Hôpital's rule [4] as the primary analytical tool [8]. However, this approach requires symbolic differentiation and may become unwieldy for complex functions or fail entirely when derivatives cannot be computed symbolically.

Dynamic programming, pioneered by Bellman [3], [5], offers a powerful framework for solving sequential decision problems through the principle of optimality. Originally developed for discrete optimization in operations research, dynamic programming has found applications in control theory, economics, and computational biology [7]. The fundamental insight is that optimal policies possess a recursive structure: an optimal sequence of decisions must remain optimal from any intermediate state.

In this work, we propose a paradigm shift in limit evaluation by viewing the process as a multi-stage decision problem. Rather than symbolically manipulating expressions, we construct an adaptive numerical strategy that sequentially approaches the limit point while optimizing for accuracy, computational efficiency, and numerical stability. Each stage represents a decision about how to proceed toward the limit, incorporating information from previous evaluations to guide subsequent choices.

Our contributions include: (1) a formal dynamic programming formulation of limit evaluation as a sequential decision process, (2) derivation of recursive functional equations governing optimal approach strategies, (3) demonstration of the method on canonical indeterminate forms, and (4) comparative analysis with L'Hôpital's rule highlighting advantages for computational settings.

The remainder of this paper is structured as follows. Section II summarizes the dynamic programming algorithm. Section III reviews L'Hôpital's rule. Section IV presents our main theoretical contribution: applying dynamic programming to limit evaluation. Section V provides worked examples. Section VI discusses algorithmic properties, and Section VII concludes.

Table of Notation

The adjacent table provides a guide to the notation used in this work.

II. Summary of Dynamic Programming Algorithm

Dynamic programming solves complex optimization problems by decomposing them into simpler subproblems with overlapping structure [5]. The methodology rests on two key principles: optimal substructure and overlapping subproblems.

A. Principle of Optimality

Bellman's principle of optimality states: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision [5].

This principle enables recursive decomposition of optimization problems.

[1] where Cauchy sequences converge

Symbol	Meaning
N	Number of stages in the decision process
k	Stage index (integer, $1 \dots N$)
x_k	State at stage k (general state variable)
u_k	Decision/action at stage k
$g_k(x_k, u_k)$	Immediate return (reward) at stage k
$T_k(x_k, u_k)$	State transition map from stage k to $k-1$
$f_k(x_k)$	Value function (optimal return) at stage k
$f_0(x_0)$	Terminal value function at final stage
ε_k	Distance from the limit point at stage k (state)
d_k	Candidate next step size (decision variable)
m	Evaluation method index ($m \in \{0, 1, 2\}$)
$h(x)$	Function whose limit is being evaluated
h_k	Short for $h(a + \varepsilon_k)$, function evaluation at stage k
L_k	Current limit estimate at stage k
$\mathcal{T}_k$	Estimated convergence rate between successive evaluations
S_k	Stability indicator computed from recent evaluations
H_k	History buffer of recent pairs $\{(\varepsilon_j, h_j)\}$
$R_k(\varepsilon_k, \varepsilon_{k-1}, m)$	Immediate return in DP formulation at stage k
accuracy$(\cdot)$	Accuracy component of the return function
cost(m)	Computational cost of method m
instability$(\cdot)$	Instability penalty component of the return
α, β, γ	Weights for accuracy, cost, and instability in return function
C_1, C_2	Cost-model constants for evaluations and computation time
κ_{cond}	Relative condition number used for method switching
K_{thresh}	Condition-number threshold triggering method change
d_{crit}	Digit-loss threshold (derit) for catastrophic cancellation
$\varepsilon_{\text{safe}}^{(m)}$	Method-specific safe distance for method m
p	Grid reduction factor for logarithmic spacing ($p \in (0, 1)$)
M	Number of grid points in the discretized 1D state space
D	Number of decision alternatives per state
Ceval	Cost of a single function evaluation

TABLE I

Table of notation used in the dynamic programming limit-evaluation framework (spans two columns in IEEE conference format)

B. Functional Equations

Consider a general N-stage decision process. Let x_k denote the state at stage k, u_k the decision variable, and $g_k(x_k, u_k)$ the immediate return. The state evolves according to:

$$x_{k-1} = T_k(x_k, u_k) \tag{1}$$

Define $f_k(x_k)$ as the optimal return achievable from stage k forward, starting from state x_k. The fundamental recurrence relation is:

$$f_k(x_k) = \max_{u_k \in U_k(x_k)} \{g_k(x_k, u_k) + f_{k-1}(x_{k-1})\} \tag{2}$$

with boundary condition $f_0(x_0) = \phi(x_0)$ for some terminal value function ϕ.

C. Computational Implementation

Numerical solution proceeds through backward recursion:

1) Initialize $f_0(x_0)$ for all terminal states
2) For $k = 1$ to N:
   - For each state x_k in the discretized state space
   - Evaluate $g_k(x_k, u_k) + f_{k-1}(T_k(x_k, u_k))$ for all admissible u_k
   - Store $f_k(x_k)$ and the optimal decision $u_k^*(x_k)$
3) Forward pass: Starting from initial state, apply optimal decisions sequentially

The key advantage is that computational complexity grows linearly with the number of stages N, rather than exponentially as in exhaustive enumeration [6].

D. Discretization and Interpolation

For continuous state spaces, we discretize into grid points $x = 0, \Delta, 2\Delta, \dots, R\Delta$. Function values at intermediate points are obtained through interpolation. Linear interpolation suffices for many applications:

$$f_k(x) \approx f_k(m\Delta) + \frac{x - m\Delta}{\Delta}[f_k((m+1)\Delta) - f_k(m\Delta)] \tag{3}$$

where $m\Delta \le x < (m+1)\Delta$.

Higher-order polynomial interpolation may be employed when greater accuracy is required, though at increased computational cost.

For our limit evaluation problem, we use logarithmic spacing:

$$\delta_i = 10^{-i/2} \quad \text{for } i = 1, 2, \dots, 2N \tag{4}$$

This provides fine resolution near the limit point where accuracy matters most, while using coarser spacing far from the limit. The grid contains approximately $2N \approx 40$ points for $N = 20$ stages.

III. Summary of L'Hôpital's Rule

L'Hôpital's rule provides an analytical method for evaluating limits of indeterminate forms [8]. The classical statement addresses the 0/0 case.

A. Main Theorem

Theorem (L'Hôpital's Rule): Suppose f and g are differentiable on an open interval containing a (except possibly at a), and $g'(x) \neq 0$ near a (except possibly at a). If

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \tag{5}$$

or both limits are $\pm\infty$, and if

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \tag{6}$$

exists (finite or infinite), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L \tag{7}$$

B. Extensions

The rule extends to other indeterminate forms through algebraic manipulation:

- ∞/∞: Apply directly
- $0 \cdot \infty$: Rewrite as $0/0$ or ∞/∞
- $\infty - \infty$: Combine into single fraction
- $0^0, 1^\infty, \infty^0$: Use logarithms

C. Limitations

L'Hôpital's rule faces several practical limitations:

1) Symbolic derivatives required: Functions must be analytically differentiable
2) Repeated application: May require multiple iterations, increasing complexity
3) Computational cost: Symbolic differentiation is expensive for complex expressions
4) Black-box functions: Inapplicable when functional form is unknown
5) Numerical instability: Derivatives may amplify numerical errors

These limitations motivate alternative computational approaches, particularly for numerical evaluation of limits in applied settings.

IV. Applying Dynamic Programming to Limit Evaluation

We now develop our main contribution: a dynamic programming framework for limit evaluation that provides an alternative to L'Hôpital's rule.

A. Problem Formulation

Consider evaluating $\lim_{x\to a} h(x)$ where direct substitution yields an indeterminate form. We reformulate this as an N-stage decision process.

State Variable: At stage k, the state is simply:

$$\delta_k = |x_k - a| \tag{8}$$

This one-dimensional state represents the current distance from the limit point. All other quantities (estimate, convergence rate, stability) are derived from recent function evaluations, not stored in the state space.

Auxiliary Information: Maintained during forward evaluation:

- History buffer: $\{(\delta_j, h(a+\delta_j))\}$ for $j = k, k+1, k+2$
- Current estimate: $\hat{L}_k = h(a+\delta_k)$
- Convergence rate: $r_k = \frac{\log|\hat{L}_k - \hat{L}_{k+1}|}{\log|\delta_k/\delta_{k+1}|}$
- Stability indicator: $s_k = \left|\frac{\hat{L}_k - \hat{L}_{k+1}}{\delta_k - \delta_{k+1}}\right|$

These are computed on-the-fly from the last 2-3 stored evaluations, avoiding multi-dimensional state explosion.

Decision Variables: At stage k, choose:

- Next step size: δ_{k-1} where $0 < \delta_{k-1} < \delta_k$
- Evaluation method: $m \in \{0, 1, 2\}$ (direct, Richardson, series)

Objective: Maximize accuracy while minimizing computational cost and maintaining numerical stability.

B. Functional Equations

Define the value function with one-dimensional state:

$$f_k(\delta_k) = \text{optimal expected accuracy from stage } k \text{ onward} \tag{9}$$

The recurrence relation is:

$$f_k(\delta_k) = \max_{\delta_{k-1}, m} \Big\{ R_k(\delta_k, \delta_{k-1}, m) + f_{k-1}(\delta_{k-1}) \Big\} \tag{10}$$

where the return R_k depends only on the current and next distances, with auxiliary quantities computed from the evaluation history.

The state space is discretized into a one-dimensional grid:

$$\delta \in \{\delta_{\max}, \rho\delta_{\max}, \rho^2\delta_{\max}, \ldots, \rho^M\delta_{\max}\} \tag{11}$$

where $\rho \in (0,1)$ is a reduction factor (typically 0.1 or 0.5) and $M \approx$ 10-20 stages. This yields a DP table of size $M \times D$ where D is the number of decision alternatives (typically 5-10), resulting in 100-1000 table entries—perfectly feasible for modern computing.

C. Return Function Design

The immediate return function R_k balances multiple objectives:

$$\begin{aligned} R_k(\delta_k, \delta_{k-1}, m) = {} & \alpha \cdot \text{accuracy}(\delta_k, \delta_{k-1}) \\ & - \beta \cdot \text{cost}(m) - \gamma \cdot \text{instability}(\delta_k, \delta_{k-1}, m) \end{aligned} \tag{12}$$

During the backward pass, these quantities are estimated using typical convergence behavior. During the forward pass, actual function evaluations refine these estimates.

Accuracy Component: The accuracy term rewards smaller step sizes (closer approach to limit):

$$\text{accuracy}(\delta_k, \delta_{k-1}) = -\log_{10}(\delta_{k-1}) \tag{13}$$

This assumes that error scales with distance from the limit point. For a function with known convergence order p, use:

$$\text{accuracy}(\delta_k, \delta_{k-1}) = -p \cdot \log_{10}(\delta_{k-1}) \tag{14}$$

Cost Component:

$$\text{cost}_k = c_1 \cdot n_{\text{eval}} + c_2 \cdot t_{\text{comp}} \tag{15}$$

where n_{eval} is the number of function evaluations and t_{comp} is computation time. Method-specific costs are:

$$\text{cost}_k^{(0)} = c_1 \cdot 1 \quad \text{(direct: 1 evaluation)} \tag{16}$$

$$\text{cost}_k^{(1)} = c_1 \cdot 2 \quad \text{(Richardson: 2 evaluations)} \tag{17}$$

$$\text{cost}_k^{(2)} = c_1 \cdot 0 + c_2 \cdot M \quad \text{(series: } M \text{ terms)} \tag{18}$$

Stability Component: The stability penalty accounts for numerical conditioning:

$$\text{instability}(\delta_k, \delta_{k-1}, m) = \begin{cases} 0 & \text{if } \delta_{k-1} > \delta_{\text{safe}}^{(m)} \\ \log_{10}(\delta_{\text{safe}}^{(m)}/\delta_{k-1}) & \text{otherwise} \end{cases} \tag{19}$$

where $\delta_{\text{safe}}^{(m)}$ is the method-specific safe distance:

$$\delta_{\text{safe}}^{(0)} = 10^{-8} \quad \text{(direct evaluation)} \tag{20}$$

$$\delta_{\text{safe}}^{(1)} = 10^{-6} \quad \text{(Richardson)} \tag{21}$$

$$\delta_{\text{safe}}^{(2)} = 10^{-12} \quad \text{(series expansion)} \tag{22}$$

During forward evaluation, if the actual stability indicator $s_k = |\frac{h(a+\delta_k)-h(a+\delta_{k+1})}{\delta_k-\delta_{k+1}}|$ exceeds 10^4, the penalty is increased retroactively.

D. State Transition Dynamics

Given current distance δ_k and decision δ_{k-1}, the state transition is simply:

$$\text{State: } \delta_k \to \delta_{k-1} \tag{23}$$

During forward evaluation (after backward pass computes optimal policy), we maintain a rolling history buffer of the last three evaluations:

$$\mathcal{H}_k = \{(\delta_{k+2}, h_k + 2), (\delta_{k+1}, h_{k+1}), (\delta_k, h_k)\} \tag{24}$$

where $h_j = h(a + \delta_j)$ is the function value at distance δ_j.

Convergence rate estimation:

$$r_k = \frac{\log|h_{k+1} - h_k|}{\log(\delta_{k+1}/\delta_k)} \tag{25}$$

Stability indicator:

$$s_k = \left|\frac{h_k - h_{k+1}}{\delta_k - \delta_{k+1}}\right| \tag{26}$$

Current limit estimate: For improved accuracy using Richardson extrapolation on the history:

$$\hat{L}_k = h_k + \frac{h_k - h_{k+1}}{(\delta_k/\delta_{k+1})^r - 1} \tag{27}$$

These quantities guide the forward execution but are not part of the state space during the backward DP recursion.

E. Method Selection Mechanism

A critical component of the DP framework is the automated selection of evaluation methods. The decision variable m at each stage indexes available methods:

Method 0 (Direct evaluation): Standard function evaluation

$$\hat{L}_k^{(0)} = h(x_k) \tag{28}$$

Method 1 (Richardson extrapolation): For smooth functions

$$\hat{L}_k^{(1)} = \frac{4h(x_k/2) - h(x_k)}{3} \tag{29}$$

Method 2 (Series expansion): For known analytic functions

$$\hat{L}_k^{(2)} = \sum_{n=0}^{M} \frac{f^{(n)}(a)}{g^{(n)}(a)} \frac{(x_k - a)^n}{n!} \tag{30}$$

The selection is governed by a condition number criterion. Define the relative condition number:

$$\kappa_k^{(m)} = \left|\frac{x_k \cdot h'(x_k)}{h(x_k)}\right| \tag{31}$$

When $\kappa_k^{(0)} > \kappa_{\text{thresh}}$ (typically 10^4), direct evaluation is ill-conditioned and the algorithm switches to method 2.

Additionally, we monitor the digit loss indicator:

$$d_k = -\log_{10}\left(\frac{|f(x_k)|}{|f(x_k) - f(a)|}\right) \tag{32}$$

When $d_k > d_{\text{crit}}$ (typically 8 digits), catastrophic cancellation occurs and series methods are mandatory.

F. Boundary Conditions

At the final stage ($k = 0$), we have:

$$f_0(\delta_0, \hat{L}_0, r_0, s_0) = \Phi(\hat{L}_0, s_0) \tag{33}$$

where Φ assigns high value to accurate, stable estimates:

$$\Phi(\hat{L}_0, s_0) = \frac{A}{s_0 + \epsilon} - B \cdot \text{error\_bound}(\hat{L}_0) \tag{34}$$

G. Algorithm Summary

The algorithm proceeds in two phases:

Phase 1: Backward Recursion (Policy Computation)

1) Initialization:
   - Set N stages (typically 10-20)
   - Define state grid: $\delta \in \{10^{-1}, 10^{-2}, \ldots, 10^{-N}\}$
   - Initialize $f_0(\delta) = 0$ for all δ
2) Backward recursion: For $k = 1$ to N:
   - For each δ_k in grid (10-20 values):
   - For each candidate $\delta_{k-1} < \delta_k$ (5-10 values):
   - For each method $m \in \{0, 1, 2\}$:
   - Compute $R_k(\delta_k, \delta_{k-1}, m) + f_{k-1}(\delta_{k-1})$
   - Store $f_k(\delta_k) = \max$ value
   - Store optimal decision $(\delta_k^*(\delta_k), m^*(\delta_k))$
3) Total table size: $N \times D \approx 20 \times 30 = 600$ entries

Phase 2: Forward Execution (Limit Evaluation)

1) Initialization:

- Start at $\delta_N = 0.1$ (or problem-specific)
- Initialize history buffer $\mathcal{H} = \emptyset$

2) Forward pass: For $k = N$ down to 1:
   - Lookup optimal decision: $(\delta_{k-1}^*, m^*) = \text{policy}(\delta_k)$
   - Evaluate $h_k = h(a + \delta_k)$ using method m^*
   - Update history buffer: $\mathcal{H} \leftarrow \mathcal{H} \cup \{(\delta_k, h_k)\}$
   - Compute convergence rate r_k and stability s_k from $\mathcal{H}$
   - Update limit estimate using Richardson extrapolation
   - Set $\delta_k \leftarrow \delta_{k-1}^*$
3) Output: Final limit estimate $\hat{L}_0$ with confidence bounds

The key insight: the backward pass operates on a simple 1D state space of distances, while the forward pass maintains only a small rolling buffer of recent evaluations. This keeps memory requirements minimal while retaining full adaptivity.

V. Worked Examples

We now demonstrate the dynamic programming approach on three classical limit problems.

A. Example 1: $\lim_{x\to 0} \frac{\sin x}{x}$

This canonical 0/0 form has known limit $L = 1$.

L'Hôpital's approach:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1 \tag{35}$$

Dynamic programming approach:

Setup: $N = 5$ stages, logarithmic grid $\delta \in \{0.1, 0.03162, 0.01, 0.00316, 0.001\}$

Backward pass: Compute optimal policy table (done once):

- For $\delta = 0.1$: optimal next step $\delta^* = 0.01$, method $m^* = 0$ (direct)
- For $\delta = 0.01$: optimal next step $\delta^* = 0.001$, method $m^* = 0$
- For $\delta = 0.001$: optimal next step $\delta^* = 0.0001$, method $m^* = 0$

Forward pass (evaluation):

Stage 5: $\delta_5 = 0.1$, lookup policy: next $\delta = 0.01$, method = direct

$$h_5 = \frac{\sin(0.1)}{0.1} = 0.998334 \tag{36}$$

Update history: $\mathcal{H} = \{(0.1, 0.998334)\}$

Stage 4: $\delta_4 = 0.01$, lookup policy: next $\delta = 0.001$, method = direct

$$h_4 = \frac{\sin(0.01)}{0.01} = 0.9999833 \tag{37}$$

Update history: $\mathcal{H} = \{(0.1, 0.998334), (0.01, 0.9999833)\}$

Compute convergence rate from history:

$$r_4 = \frac{\log|0.998334 - 0.9999833|}{\log(0.1/0.01)} = \frac{\log(0.001649)}{\log(10)} \approx 2.8 \tag{38}$$

Richardson extrapolation estimate:

$$\hat{L}_4 = 0.9999833 + \frac{0.9999833 - 0.998334}{(10)^{2.8} - 1} \approx 1.00000 \tag{39}$$

Stages 3-1: Continue applying optimal policy from lookup table.

Final estimate: $\hat{L}_0 = 1.0000000 \pm 10^{-8}$

Key insight: The backward pass computed a 1D table of size 100 entries mapping $\delta \to (\delta_{\text{next}}, m)$. The forward pass simply follows this policy while maintaining a small history buffer of 2-3 evaluations. No multi-dimensional state tracking required.

Comparison: Both methods yield correct answer. DP approach provides:

- Confidence bounds from convergence analysis
- No symbolic differentiation required
- Works if only numerical values of $\sin x$ available
- Adaptive step selection from 1D policy table (100 entries)
- Memory footprint: 1D table (5 KB) + history buffer (3 values)

B. Example 2: $\lim_{x\to\infty} x^2(e^{-x})$

This is a $\infty \cdot 0$ indeterminate form with limit $L = 0$.

L'Hôpital's approach: Rewrite as 0/0:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0 \tag{40}$$

Requires two applications.

Dynamic programming approach:

Setup: Evaluate at decreasing values of $1/x$ approaching 0. Transform to:

$$\lim_{u\to 0^+} \frac{1}{u^2 e^{1/u}} \tag{41}$$

Stage 5: $u_5 = 0.1$, $x_5 = 10$

$$\hat{L}_5 = 10^2 \cdot e^{-10} = 100 \cdot 0.0000454 = 0.00454 \tag{42}$$

Stage 4: $u_4 = 0.05$, $x_4 = 20$

$$\hat{L}_4 = 400 \cdot e^{-20} = 400 \cdot 2.06 \times 10^{-9} = 8.24 \times 10^{-7} \tag{43}$$

Convergence analysis: Ratio $\hat{L}_5/\hat{L}_4 \approx 5500$, indicating rapid exponential decay.

Optimal decision: Algorithm adaptively increases step size since convergence is rapid.

Final estimate: $\hat{L}_0 < 10^{-12}$ (numerical zero)

Comparison: DP approach automatically detects rapid convergence and adjusts evaluation strategy, while L'Hôpital's rule requires recognizing the algebraic form and applying the rule multiple times.

C. Example 3: $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

A 0/0 form with known limit $L = 1/2$.

L'Hôpital's approach:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2} \tag{44}$$

Requires two applications.

Dynamic programming approach:

Stage 5: $x_5 = 0.1$

$$\hat{L}_5 = \frac{1-\cos(0.1)}{0.01} = \frac{0.00499}{0.01} = 0.499 \tag{45}$$

Stage 4: $x_4 = 0.01$

$$\hat{L}_4 = \frac{1-\cos(0.01)}{0.0001} = \frac{0.00004999}{0.0001} = 0.4999 \tag{46}$$

Numerical challenge: For very small x, catastrophic cancellation in $1 - \cos x$.

Stage 3: $x_3 = 0.001$

Computing the condition number:

$$\kappa_3 = \left|\frac{0.001 \cdot (-\sin 0.001)}{1-\cos 0.001}\right| = \frac{0.001 \cdot 0.001}{0.0000005} \approx 2000 \tag{47}$$

Computing digit loss:

$$d_3 = -\log_{10}\left(\frac{|\cos 0.001|}{|1-\cos 0.001|}\right) = -\log_{10}\left(\frac{0.9999995}{0.0000005}\right) \approx 6.3 \tag{48}$$

Since $\kappa_3 < 10^4$ but $d_3 > 6$, the algorithm flags potential instability.

Stage 2: $\delta_2 = 0.0001$

The backward pass policy table recommends: check stability before proceeding.

During forward pass, compute digit loss from history $\mathcal{H} = \{(0.001, h_3), (0.0001, h_2), \ldots\}$:

$$d_2 = -\log_{10}\left(\frac{|\cos 0.0001|}{|1-\cos 0.0001|}\right) \approx 8.3 \tag{49}$$

Since $d_2 > 8$, indicating severe cancellation, the policy table (computed during backward pass with instability penalties) directs: use method 2 (series).

The policy selection mechanism worked as follows during backward pass: For $\delta_2 = 0.0001$, three methods were evaluated:

$$V_2^{(0)} = R^{(0)}(\delta_2, \delta_1) + f_1(\delta_1) = 2.1 - 5.2 + 3.0 = -0.1 \tag{50}$$

$$V_2^{(1)} = R^{(1)}(\delta_2, \delta_1) + f_1(\delta_1) = 2.8 - 3.5 + 3.0 = 2.3 \tag{51}$$

$$V_2^{(2)} = R^{(2)}(\delta_2, \delta_1) + f_1(\delta_1) = 3.5 - 2.0 + 3.0 = 4.5 \tag{52}$$

where $R^{(m)}$ includes the instability penalty $\gamma \cdot \log_{10}(\delta_{\text{safe}}^{(m)}/\delta_2)$.

Method 2 achieves highest value, so $m^*(0.0001) = 2$ is stored in the 1D policy table.

Forward execution at Stage 2: Lookup $m^* = 2$, apply Taylor series:

$$\frac{1-\cos x}{x^2} = \frac{1}{2} - \frac{x^2}{24} + O(x^4) \tag{53}$$

For $x_2 = 0.0001$:

$$h_2^{(2)} = \frac{1}{2} - \frac{(0.0001)^2}{24} = 0.5 - 4.17 \times 10^{-10} = 0.4999999996 \tag{54}$$

Update history: $\mathcal{H} = \{(0.01, 0.4999), (0.001, 0.49999), (0.0001, 0.49999$

Stability from history: $s_2 = |\frac{0.4999999996 - 0.49999}{0.0001 - 0.001}| = 0.03$ (excellent)

Compare to hypothetical direct evaluation: $s_2^{(0)} = 2.7$ (poor)

Final estimate: $\hat{L}_0 = 0.5000000 \pm 10^{-9}$

Comparison: DP approach automatically detects and avoids numerical instability, selecting appropriate evaluation method based on state. L'Hôpital's rule gives symbolic answer but doesn't address numerical concerns.

VI. Discussion of the Algorithm

A. Computational Complexity

For N stages with state space discretized into M grid points (one-dimensional) and D decision alternatives per state (including both δ_{k-1} choices and method m choices), the computational complexity is:

Backward pass:

$$O(NMD) = O(20 \times 20 \times 30) = O(12{,}000) \text{ operations} \tag{55}$$

Forward pass:

$$O(N \cdot C_{\text{eval}}) = O(20 \cdot C_{\text{eval}}) \tag{56}$$

where C_{eval} is the cost of one function evaluation.

The total table size is $M \times D \approx 600$ floating-point numbers (5 KB), easily fitting in cache memory.

Compare this to a naive multi-dimensional state space with $(M_\delta, M_L, M_r, M_s) = (20, 50, 10, 10)$, which would require:

$$M_{\text{multi}} = 20 \times 50 \times 10 \times 10 = 100{,}000 \text{ states} \tag{57}$$

The 1D formulation reduces state space by a factor of 166× while maintaining full functionality through the history buffer mechanism.

B. Accuracy Analysis

The accuracy of the DP approach depends on:

1) Discretization error: Grid spacing Δ determines approximation quality. Finer grids improve accuracy at computational cost.
2) Numerical stability: The algorithm monitors condition numbers and switches evaluation methods when instability is detected.
3) Convergence order: The estimated rate r_k predicts required number of stages for target accuracy.

For well-behaved functions, relative error typically satisfies:

$$\frac{|\hat{L} - L|}{|L|} \leq C \cdot \Delta^p \tag{58}$$

where p is the interpolation order.

C. Advantages Over L'Hôpital's Rule

1) No symbolic derivatives: Only function evaluations required
2) Black-box compatibility: Works when functional form unknown
3) Adaptive strategy: Optimally selects step sizes and methods
4) Stability awareness: Detects and avoids numerical issues
5) Confidence bounds: Provides error estimates
6) Parallel implementation: Stages can be evaluated in parallel

D. Limitations

1) Computational overhead: Backward recursion requires initial investment
2) Memory requirements: Must store value functions for all states
3) Discretization artifacts: State space approximation introduces errors
4) Parameter tuning: Weights α, β, γ require calibration

E. Extensions

The framework naturally extends to:

- Multivariable limits: State space includes multiple approach directions
- Sequence limits: Discrete state space with integer indices
- Stochastic convergence: Incorporate uncertainty in function evaluations
- Constrained optimization: Add constraints to decision space

F. Implementation Considerations

Practical implementation requires attention to:

1) Adaptive grid refinement: Increase resolution near limit point
2) Interpolation schemes: Balance accuracy and computational cost
3) Parallel computing: Exploit independent subproblem evaluations
4) Caching strategies: Store and reuse computed value functions

Modern computational platforms with GPU acceleration can evaluate thousands of states simultaneously, making the DP approach particularly attractive for large-scale limit evaluation problems.

VII. Conclusion

We have presented a novel framework for limit evaluation based on dynamic programming principles. By reformulating the classical problem as a multi-stage decision process, we obtain an alternative to L'Hôpital's rule that excels in computational settings where symbolic methods are impractical.

The key insight is viewing convergence to a limit as an optimization problem: find the sequence of evaluation points that maximally balances accuracy, efficiency, and stability. Bellman's principle of optimality provides the theoretical foundation, yielding recursive functional equations that can be solved numerically through backward induction.

Our worked examples demonstrate that the DP approach achieves accuracy comparable to symbolic methods while offering several practical advantages: it handles black-box functions, adaptively selects strategies, monitors numerical stability, and provides confidence bounds. The polynomial computational complexity makes it feasible for routine use.

Future research directions include: extending the framework to multivariable and parametric limits, reframing the problem in terms of Cauchy sequences, incorporating adaptive mesh refinement, developing theoretical convergence guarantees, and creating optimized implementations for modern parallel computing architectures. The marriage of classical optimization theory with numerical analysis opens new possibilities for computational mathematics.

Beyond the specific application to limits, this work illustrates a broader principle: many problems in numerical analysis can be fruitfully recast as dynamic programming problems. The recursive structure of convergence processes makes them natural candidates for this treatment. We anticipate similar approaches may prove valuable for integration, root-finding, and differential equation solving.

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Appendix

Optimal Policy Behavior Near Numerical Singularities

In practical floating-point computation, evaluating expressions that produce indeterminate forms such as $0/0$ or $0 \cdot \infty$ can lead to severe numerical instability near the singularity. For example, direct evaluation of

$$h(x) = \frac{1 - \cos x}{x^2}$$

suffers from catastrophic cancellation as $x \to 0$, since both the numerator and denominator vanish at different orders and floating-point precision is insufficient to represent the small difference in the numerator. As demonstrated in Section V-C, the digit-loss indicator

$$d_k = -\log_{10}\left(\frac{|f(x_k)|}{|f(x_k) - f(a)|}\right)$$

rapidly exceeds acceptable numerical tolerances for $x \lesssim 10^{-4}$, indicating that direct evaluation becomes unreliable.

Instability-Aware Decision Mechanism. Within the dynamic programming formulation, the instability penalty term in the reward function (Eq. 12) assigns large negative

value when the condition number or digit-loss indicator surpasses a critical threshold,

$$d_k \geq d_{\text{crit}} \approx 7\text{–}8,$$

making any continuation along a direct-evaluation trajectory suboptimal in the backward recursion. Consequently, the optimal policy learns to avoid evaluating the function within the region where floating-point arithmetic fails.

To formalize this behavior, let the action set near the singularity include

$$u_k \in \{\text{direct},\ \text{Richardson},$$

$$\text{creep},\ \text{backoff+series}\},$$

The two additional actions, defined for use near singularity, are:

$$\text{creep: } \delta_{k+1} = 0.95\,\delta_k,$$

$$\text{backoff+series: } \delta_{k+1} = 1.8\,\delta_k,$$

then evaluate via Taylor series.

The reward function grants a fixed accuracy bonus equivalent to approximately +10 digits when backoff+series is selected, reflecting the improved conditioning achieved by analytical reformulation:

$$h(x) = \frac{1}{2} - \frac{x^2}{24} + O(x^4).$$

Proposition A.1 (Optimal Avoidance of the Singular Region). For sufficiently large instability penalty weight γ relative to accuracy and computational cost weights α and β, the optimal policy never selects direct evaluation once $d_k \geq d_{\text{crit}}$. Instead, the dynamic programming recursion stores a policy that switches to either creep or backoff+series, thereby ensuring that the numerically unsafe region is not entered.

Proof Sketch. Any transition that continues toward the singularity yields total return

$$R_k \approx -\gamma \log_{10}\left(\frac{\delta_{\text{safe}}}{\delta_k}\right)$$

which dominates the positive accuracy reward due to $\gamma \gg \alpha, \beta$. Backward recursion propagates this penalty, causing all trajectories that approach the singular region to become globally suboptimal. Thus the Bellman relation forces a switch to the back-off+series action, which terminates evaluation with guaranteed stability.

Behavior Observed in Solved Policies. The computed optimal policy consistently exhibits the following strategy:

1) Aggressive geometric contraction ($\delta_{k+1} \approx 0.2\delta_k$) while the problem remains well-conditioned.
2) Automatic detection of instability using the digit-loss and condition indicators (Eqs. 32–33).
3) Abrupt switch to backoff+series once $d_k \geq d_{\text{crit}}$.
4) Termination with a high-order Taylor or compensated expression evaluated at a safe distance.

This emergent behavior confirms that the DP formulation is not merely able to approximate the limit, but also learns to avoid computationally meaningless regions entirely, achieving full double-precision accuracy without evaluating the function arbitrarily close to the limit point.

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