# On the Hodgkin-Huxley Equation on a Curve

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**Abstract**

In this note, the author takes a few steps towards setting up the Hodgkin-Huxley equation, but along a curve. The new formulation has potentially high relevance for realistic modeling of tracts of ephaptically interacting axons.

We begin by recalling the Hodgkin Huxley equation [1, 2, 3]in the following form,

$$\frac{\partial V}{\partial t} = \kappa_1 \frac{\partial^2 V}{\partial x^2} - \kappa_2 V. \tag{1}$$

Next, we introduce a pair of curves in space and time, $(g(x), h(t))$, along which an axon lies according to the rule,

$$x_{pos} = g(x) \tag{2}$$

and

$$t_{pos} = h(t) \tag{3}$$

where $x_{pos}$ is the spatial location along the axon and $t_{pos}$ is the temporal location at which that spatial location is being considered. Our task is to find the new version of Equation (1). Using the chain rule for differentiation,

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial g} \cdot \frac{\partial g}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t} = 0 + \frac{\partial V}{\partial h} \cdot \frac{\partial h}{\partial t} \tag{4}$$

and

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial g} \cdot \frac{\partial g}{\partial x} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial x} = \frac{\partial V}{\partial g} \cdot \frac{\partial g}{\partial x} + 0. \tag{5}$$

Thus,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}[\frac{\partial V}{\partial g} \cdot \frac{\partial g}{\partial x}] = \frac{\partial V}{\partial g} \cdot \frac{\partial^2 g}{\partial x^2} + \frac{\partial}{\partial x}[\frac{\partial V}{\partial g}] \cdot \frac{\partial g}{\partial x}. \tag{6}$$

Thus,

$$\frac{\partial V}{\partial h} \cdot \frac{\partial h}{\partial t} = \kappa_1 [\frac{\partial V}{\partial g} \cdot \frac{\partial^2 g}{\partial x^2} + \frac{\partial}{\partial x}[\frac{\partial V}{\partial g}] \cdot \frac{\partial g}{\partial x}] - \kappa_2 V. \tag{7}$$

Consider the case that

$$g(x) = \cos x \tag{8}$$

and

$$h(t) = t. \tag{9}$$

That is, the axon "snakes" along a cosinusoidal path. In this case, the Hodgkin Huxley equation becomes,

$$\frac{\partial V}{\partial h} = -\kappa_1 [\cos x \cdot \frac{\partial V}{\partial g} + \sin x \cdot \frac{\partial}{\partial x}[\frac{\partial V}{\partial g}]] - \kappa_2 V. \tag{10}$$

The study of ephaptic coupling in tracts of such axons [4], in three dimensions, with tortuosity [5] and so on, becomes an entirely different, but related, line of study, with potential high relevance to realistic modeling. In particular, the tortuosity formulation can be made much richer when each infinitesimal segment is of this form.

# References


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