On a Work by Prof. Sergio Servetto

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Abstract

In this note, the authors re-interpret Theorem 1 of Servetto using Theorem 3 of Shannon and relate it to Sahai's anytime repeated pulse position modulation tree code and anytime information theory in general. Under numerous assumptions, they find that zero-error network information flow with correlated sources on tree graphs seems like a simultaneous and similar approach to the problem of reliable transmission of data, as does anytime information theory.

This work is dedicated to the memory of Prof. Sergio Servetto, a great teacher and mentor.

Consider Theorem 1 of Servetto [1]. It tells us an upper bound on the conditional entropy of two sets of random variables split by a cut in a graph. One side of the cut contains the destination node and the other side is the source side. It says that for any such cut, the conditional entropy of the source side random variables given the sink side random variables must be less than the sum of the link capacities where only those links which cross the cut are considered. If we think in zero error terms, and suppose that all these link capacities are such that we can write down their zero error capacity C_0 by Theorem 3 of Shannon [2], then the theorem states that the conditional entropy (aforesaid) is less than the log of the product of all the "reduced" alphabet sizes. When we are taking the product, it is like a counting argument. Thus we have counted out or enumerated all the possible ways by which an error-free transmission can take place across the cut. Thus Theorem 1 of Servetto says that the source side can be sent to the sink side if and only if the conditional surprise on the source side given the sink side is less than the log of the number of ways one can basically glide (unperturbed) to the sink side. This latter is nothing but the number of reliable (error free) bits one can send across the cut.

Next consider an anytime RPPM tree code as in Sahai [3]. As the tree grows, the bits are repeated and so become more and more reliable. Think of a cut as a line separating two adjacent "levels" of the tree. On one side (the source side) there are incoming bits and on the other side (the destination side) there are outgoing bits. The number of branches grows as the tree grows, in fact doubling every level. If the problem is framed as the transmission of the source side bits (or random variables) to the sink side on this "graph," then we see that, applying Theorem 1 of Servetto, the conditional entropy of the source side bits given the sink side bits must be less than the number of error free bits one can send across the cut from the prior level to the next level. This must hold for every possible cut. So whenever we "finalize" the stream, the conditional entropy is upper bounded by log of half the number of branches on the sink side of the cut. The later we finalize the stream, the looser this upper bound becomes, but the number of bits on the source side also grows. In contrast, Sahai's anytime exponent upper bound grows smaller the later we finalize the stream, i.e., becomes tighter. But we may be comparing apples and oranges, because conditional entropy is bounded in [1] whereas an error probability is bounded in [3]. This last point rescues the argument, for as a probability goes smaller, the corresponding entropy-like term $-p \log p$ grows larger. Thus both bounds are exploding more, the later we finalize the stream.

In conclusion, for tree-like networks, there are deep connections between anytime reliable transmission and Shannon reliable transmission as discussed by Servetto. Thus the cursory appearance that anytime information theory generalizes Shannon's formulation may not be factual, at least for some settings. They may be addressing a similar problem in different ways. Future work should quantify the difference between the Servetto "exponent" and the anytime exponent. It appears that the latter, being equal to the Shannon exponent for the infinite bandwidth AWGN channel (see Section 3.2 in [3]), would agree with the Shannon exponent derived in Prof. Servetto's paper, at least in principle, and on our impression.

References

[1] J. Barros and S.D. Servetto. Network information flow with correlated sources. *IEEE Transactions on Information Theory*, 52(1):155–170, 2006.

[2] C. Shannon. The zero error capacity of a noisy channel. *IRE Transactions on Information Theory*, 2(3):8–19, 1956.

[3] Anant Sahai. Anytime coding on the infinite bandwidth awgn channel: A sequential semi-orthogonal code. In *Proc. 2005 Conf. Inf. Sci. Syst.* Citeseer, 2005.