Mass as a State: Impact on the Heisenberg Equations of Motion

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Abstract

In this note, the authors present details of how viewing mass as an observable impacts the Hesienberg equations of motion of the system and its energy.

Herein the authors present detailed considerations on treating mass as an observable. In the context where we treat mass as an operator, we can derive the Heisenberg equations of motion to understand how the position and momentum operators evolve over time. This involves reinterpreting the Heisenberg framework, in which the time evolution of observables (operators) depends on the Hamiltonian, incorporating the mass operator $\hat{m}$ as part of the dynamics.

Background: Heisenberg Equation of Motion

In the Heisenberg picture, the equation of motion [1, 2] for an operator $\hat{A}$ is given by:

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}$$

where $\hat{H}$ is the Hamiltonian of the system. This equation describes how observables change with time in the Heisenberg picture, where operators evolve in time while the quantum states remain fixed.

Hamiltonian with Mass as an Operator

If we treat mass m as an operator $\hat{m}$, the Hamiltonian for a free particle [3] would become:

$$\hat{H} = \frac{\hat{p}^2}{2\hat{m}} + V(\hat{x})$$

where:

- $\hat{p}$ is the momentum operator,
- $\hat{x}$ is the position operator,
- $V(\hat{x})$ is a potential energy that may or may not depend on the mass operator.

Equations of Motion for Position and Momentum Operators

Now, let's apply the Heisenberg equation of motion to the position and momentum operators, $\hat{x}$ and $\hat{p}$, under this Hamiltonian with the mass operator $\hat{m}$.

Position Operator $\hat{x}$

For the position operator $\hat{x}$, we have:

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{x}]$$

Since $\hat{H} = \frac{\hat{p}^2}{2\hat{m}} + V(\hat{x})$, let's calculate the commutator $[\hat{H}, \hat{x}]$:

1. First, we use that $[\hat{p}^2, \hat{x}] = \hat{p}[\hat{p}, \hat{x}] + [\hat{p}, \hat{x}]\hat{p} = -2i\hbar\hat{p}$, given that $[\hat{p}, \hat{x}] = -i\hbar$.
2. Thus:

$$[\hat{H}, \hat{x}] = \frac{1}{2\hat{m}} \cdot (-2i\hbar\hat{p}) = -\frac{i\hbar\hat{p}}{\hat{m}}$$

So,

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{\hat{m}}$$

This is analogous to the classical expression for velocity, but now $\hat{m}$ is an operator, which could vary depending on the system's state.

Momentum Operator $\hat{p}$

Note that $\hat{m}$ is an operator, which should be borne in mind when evaluating the commutation relation involving $\frac{\hat{p}^2}{2\hat{m}}$ and $\hat{p}$. When $\hat{m}$ is treated as an operator, it no longer simply commutes with other terms in the Hamiltonian, and we must carefully evaluate the commutator $\left[\frac{\hat{p}^2}{2\hat{m}}, \hat{p}\right]$ with this in mind.

Correct Calculation of $\left[\frac{\hat{p}^2}{2\hat{m}}, \hat{p}\right]$

Our Hamiltonian in the context where $\hat{m}$ is treated as an operator is:

$$\hat{H} = \frac{\hat{p}^2}{2\hat{m}} + V(\hat{x})$$

We are interested in finding $\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}]$, which includes the term $\left[\frac{\hat{p}^2}{2\hat{m}}, \hat{p}\right]$.

- Expand the Commutator $\left[\frac{\hat{p}^2}{2\hat{m}}, \hat{p}\right]$

Using the product rule for commutators, we have:

$$\left[\frac{\hat{p}^2}{2\hat{m}}, \hat{p}\right] = \frac{1}{2}\left([\hat{p}^2, \hat{p}]\frac{1}{\hat{m}} + \hat{p}^2\left[\frac{1}{\hat{m}}, \hat{p}\right]\right)$$

Next, we evaluate each term separately.

- Evaluate $[\hat{p}^2, \hat{p}]\frac{1}{\hat{m}}$:

Since $[\hat{p}^2, \hat{p}] = 0$ (momentum commutes with itself), this term is zero.

$$[\hat{p}^2, \hat{p}] = 0$$

- Evaluate $\hat{p}^2\left[\frac{1}{\hat{m}}, \hat{p}\right]$:

To proceed, we need to determine $\left[\frac{1}{\hat{m}}, \hat{p}\right]$. Let's assume that $\hat{m}$ does not commute with $\hat{p}$, implying that $\hat{m}$ and $\hat{p}$ have a non-trivial commutation relationship.

If we suppose that $[\hat{m}, \hat{p}] = i\hbar\hat{f}(\hat{x})$ (for some function $\hat{f}(\hat{x})$), then we can compute $\left[\frac{1}{\hat{m}}, \hat{p}\right]$ using the identity (which is trivial to verify):

$$\left[\frac{1}{\hat{m}}, \hat{p}\right] = -\frac{1}{\hat{m}}\left[\hat{m}, \hat{p}\right]\frac{1}{\hat{m}}$$

Substituting $[\hat{m}, \hat{p}] = i\hbar\hat{f}(\hat{x})$, we get:

$$\left[\frac{1}{\hat{m}}, \hat{p}\right] = -\frac{i\hbar}{\hat{m}}\hat{f}(\hat{x})\frac{1}{\hat{m}}$$

So, our expression for $\hat{p}^2\left[\frac{1}{\hat{m}}, \hat{p}\right]$ becomes:

$$\hat{p}^2\left[\frac{1}{\hat{m}}, \hat{p}\right] = -i\hbar\hat{p}^2\frac{\hat{f}(\hat{x})}{\hat{m}^2}$$

Final Form of the Heisenberg Equation for $\hat{p}$

Now, putting it all together, we have:

$$\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -\frac{dV}{dx} - \frac{\hat{p}^2\hat{f}(\hat{x})}{\hat{m}^2}$$

where $\hat{f}(\hat{x})$ represents the commutation factor between $\hat{m}$ and $\hat{p}$. This additional term reflects the effect of treating mass as a dynamic operator, potentially leading to additional "forces" depending on the functional form of $\hat{f}(\hat{x})$.

This adjustment shows that introducing $\hat{m}$ as an operator fundamentally changes the dynamics of $\hat{p}$, adding a non-trivial term to the equation of motion that is absent in standard quantum mechanics. This term depends on the specific commutation relationship between $\hat{m}$ and $\hat{p}$, which could vary based on the system or theoretical context.

Summary of Heisenberg Equations of Motion with Mass as an Operator

1. Position Operator:

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{\hat{m}}$$

This implies that the velocity depends inversely on the mass operator $\hat{m}$, making it a dynamic observable if the system allows for mass variations.

2. Momentum Operator:

$$\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -\frac{dV}{dx} - \frac{\hat{p}^2 \hat{f}(\hat{x})}{\hat{m}^2}$$

Figure 1 illustrates this equation graphically.

These equations illustrate how allowing mass to be an operator introduces new dynamics in the Heisenberg picture, particularly in the equation for velocity, which becomes a function of the mass operator. This formalism could be interesting in speculative theories where mass is dynamic, such as certain quantum gravity theories or situations with effective masses that vary based on interaction or energy scales.

References

[1] Werner Heisenberg. The actual content of quantum theoretical kinematics and mechanics. Technical report, 1983.

[2] Tomoi Koide and Masahiro Maruyama. A new expansion of the heisenberg equation of motion with projection operator. *Progress of Theoretical Physics*, 104(3):575–594, 2000.

[3] B Supriadi, T Prihandono, V Rizqiyah, ZR Ridlo, N Faroh, and S Andika. Angular momentum operator commutator against position and hamiltonian of a free particle. In *Journal of Physics: Conference Series*, volume 1211, page 012051. IOP Publishing, 2019.

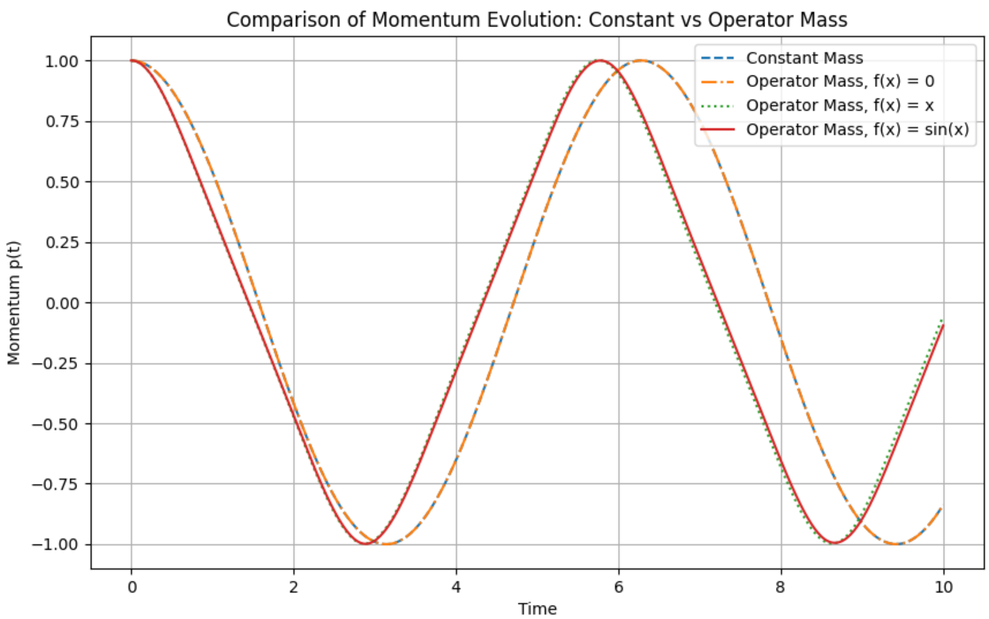

Figure 1: Correction to the Heisenberg Equations of Motion for Momentum Evolution with the New Mass Operator. Observe that when mass is an operator, the momentum oscillates at a higher frequency. In other words, system energy is changed towards the higher side.