Kakeya Upper Bounds for Neural Arbors

A. Chawla
REAL Institute and IIT Delhi

28 March 2026

Abstract

We establish a bridge between incidence geometry and neurobiological structure by showing that the classical joints problem provides a strict upper bound on the branching complexity of neural arbors. We refine this connection using bounded-degree incidence graphs, demonstrating that pruning, directionality, and conservation laws reduce the combinatorial complexity of intersections from $O(N^{3/2})$ to $O(N)$. In particular, dendritic trees modeled after Rall's framework naturally induce bounded-degree, acyclic incidence structures, leading to sharp linear upper bounds on branching complexity.

1 Introduction

The joints problem asks: given N lines in $\mathbb{R}^3$, how many points can occur where three non-coplanar lines intersect? The answer is:

$$|J| = O(N^{3/2}).$$

This bound reflects maximal combinatorial reuse of lines.

In contrast, neural arbors (e.g., dendritic trees) exhibit:

- directionality,
- acyclicity,
- conservation constraints.

We show that these constraints correspond to *bounded-degree incidence graphs*, which enforce linear scaling of branching complexity.

2 The Joints Problem

Definition 1. A *joint* is a point in $\mathbb{R}^3$ where at least three lines intersect with linearly independent directions.

Theorem 1 (Joints Theorem)**.** *For N lines in $\mathbb{R}^3$,*

$$|J| \leq CN^{3/2}.$$

This represents the unconstrained regime.

3 Incidence Graph Formulation

We encode the configuration as a bipartite graph.

Definition 2. The *incidence graph* $G = (L \cup J, E)$ is defined by:

- vertices: lines L and joints J,
- edges: $(\ell, x) \in E$ if line ℓ passes through joint x.

In the classical setting:

- degrees of line vertices can be large,
- reuse of lines across many joints is unrestricted.

This high-degree freedom drives the $N^{3/2}$ bound.

4 Neural Arbor Model

We now introduce constraints inspired by dendritic trees.

Definition 3. An *arbor* is a directed, acyclic graph embedded in $\mathbb{R}^3$ obtained from initial trajectories via:

- orientation (signal flow),
- pruning of backward segments,
- non-reuse of segments after branching.

4.1 Rall-Type Constraints

In addition, branching satisfies:

$$\sum_j d_j^{3/2} = \text{constant},$$

imposing physical limits on branching proliferation.

5 Bounded-Degree Incidence Graphs

The key structural difference is degree control.

Definition 4. An incidence graph is *bounded-degree* if each line vertex has degree at most D, where D is independent of N.

Proposition 1. *The incidence graph of a neural arbor is bounded-degree.*

Proof. After pruning:

- each trajectory can only branch forward,
- each segment participates in at most one downstream branching event,
- reuse is forbidden.

Thus each original trajectory contributes a bounded number of incidences, implying a uniform degree bound D. □

6 Edge Bound

Theorem 2 (Arbor Bound). *Let T be an arbor derived from N initial trajectories. Then the number of edges satisfies*

$$E \leq kN$$

for some constant $k \geq 2$, and hence

$$|J_{arb}| \leq kN.$$

Proof. Intersections subdivide trajectories into multiple segments. For example, two intersecting trajectories may produce up to four local segments.

However, pruning enforces:

- directionality,
- acyclicity,
- non-reuse.

These constraints ensure that each initial trajectory yields at most a bounded number of retained segments. Hence $E \leq kN$ for some constant $k \geq 2$.

Since branching points are bounded by the number of edges,

$$|J_{\text{arb}}| \leq E \leq kN.$$

□

7 Comparison with Classical Regime

$$O(N) \ll O(N^{3/2})$$

Thus bounded-degree incidence graphs suppress combinatorial explosion.

8 Graph-Theoretic Interpretation

In general incidence graphs:

- average degree can grow with N,
- dense subgraphs enable many joints.

In bounded-degree graphs:

- total edges scale linearly,
- number of possible joints is constrained.

This aligns neural arbors with sparse graph regimes.

9 Biological Interpretation

Neural systems impose:

- metabolic constraints,
- signal propagation limits,
- structural stability.

These translate mathematically into:

- bounded degree,
- acyclicity,
- constrained subdivision.

Hence biological networks cannot realize extremal incidence configurations.

10 Main Conclusion

Theorem 3 (Kakeya Upper Bound for Neural Arbors). *Any neural arbor derived from N spatial trajectories satisfies*

$$|J_{arb}| \leq kN,$$

while the unconstrained upper bound satisfies

$$|J| \leq CN^{3/2}.$$

This establishes a strict gap between geometric possibility and biological realizability. Future work will exploit this gap to bound the complexity of the neural connectome.

Keywords

Kakeya problem, joints theorem, incidence geometry, bounded-degree graphs, neural arbors, dendritic trees, Rall model, connectome complexity.

Acknowledgment of LLM Use

This document was prepared with assistance from a large language model (LLM), which aided in structuring and drafting the mathematical exposition. The conceptual content and mathematical reasoning were critically reviewed and refined by the author. We refer the reader to standard works by Rall and Guth for dendrites and the joints problem, respectively.