INTEGRATION WITH RESPECT TO SIC-POVMS: A GENERALIZED VIEW

A. CHAWLA

Abstract. In this note the authors propose to view the Mitter-Young theory of integration with respect to operator valued measures, as the foundation for the theory of SIC-POVMs, proposed by Renes et. al.

Consider the definition of a random variable's expected value. We have,

$$E[\mathbb{X}] = \int x f_{\mathbb{X}}(x) dx \tag{0.1}$$

in standard notation. The alternate notation is,

$$E[\mathbb{X}] = \int x df. \tag{0.2}$$

This highlights the fact that f is the measure with respect to which the integration is being performed. Here $f_{\mathbb{X}}(x)$ is a real-valued measure. In quantum physics, the measures involved are operator-valued.

Specifically, Mitter and Young [1] have developed an entire theory of integration [4] with respect to operator-valued measures. The present note is meant simply to observe that this theory can be viewed as foundational to the case of symmetric-informationally-complete quantum measurements, via quantum measure theory [5]. The SIC theory is developed in [3] and as an application, it is mentioned that it has overlaps with the development of certain classes of codes. A view of SIC-POVMs as being founded on the Mitter-Young theory, would therefore be useful in computing the properties of quantum code words and in general doing coding, decoding and transmission using those code words, apart from estimation and detection which are explicitly discussed in [1].

Furthermore, the mathematical grounding of the Renes et. al. work in terms of Mitter-Young would serve to generalize it. For instance, Section III in [3] refers to the SIC-POVMs as forming an eigenset of some group action U_g. Under the Mitter-Young framework, Section 3, where a POVM is characterized as a linear operator from $C_0(S, X)$ to Z^*, and an injection is established, we could ask how the injection specializes to the eigenspace formed by the SIC-POVMs. Are there special features that can be identified for that particular subspace? Such investigations are not without precedent. For example, the work of Lidar et. al. [2], more than two decades old, showed how decoherence-free-subspaces can protect quantum information.

To conclude, the Renes et. al. SIC-POVM work, if brought into the fold of the mathematically foundational Mitter-Young work, may serve a broader set of domains of application.

References

[1] Mitter, S.K. and Young, S.K., 1984. Integration with respect to operator-valued measures with applications to quantum estimation theory. *Annali di Matematica Pura ed Applicata*, 137(1), pp.1–39.

[2] D. A. Lidar, I. L. Chuang, and K. B. Whaley, "Decoherence-free subspaces for quantum computation," *Phys. Rev. Lett.*, vol. 81, no. 12, pp. 2594–2597, 1998, doi: 10.1103/PhysRevLett.81.2594.

[3] Renes, J.M., Blume-Kohout, R., Scott, A.J. and Caves, C.M., 2004. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45(6), pp.2171–2180.

[4] Stroock, D.W., 2014. *An introduction to the theory of integration.* Cambridge: Cambridge University Press.

[5] Hamhalter, J., 2003. *Quantum measure theory.* Fundamenta Mathematicae, vol. 180. Dordrecht: Kluwer Academic Publishers.

REAL Institute, Gurugram
India
Email address, A. Chawla: editor@realinstitute.live
URL: https://www.realinstitute.live