Geometric Relativity of Action Potentials: a Mathematical Approach

Aman Chawla
Department of Computer Science and Engineering
Indian Instiute of Technology
Delhi, INDIA

Abstract

Background: Axon tracts containing a large number of axons, and relatively insulated from the rest of the nervous system, can be modeled to have current-mediated coupling amongst the constituent axons. This is known as ephaptic coupling. **Results:** In this paper, the authors provide a discussion which shows that there exist restrictions on geometries of axon tracts such that they would allow synchronization between axons in the tract. The restrictive conditions are presented succinctly. We also discuss a certain equivalence between the average model and the universal model for axon coordinates. **Conclusions:** Information is normallly thought of as being transferred longitudinally along axons. However, synchronization enables lateral information flow. The paper's results therefore have implications for understanding lateral information flow in axon tracts and thus the paper enables construction of new information-theoretic models of ephaptic channels.

Keywords

ephaptic coupling, geometric calculus, synchronization condition

1 Introduction and Background

Synchronization is a widespread phenomenon in nature and in the brain. We look at synchronization in axons. As per Freeman [1],

> Brains rely less on tight phase-locking of small numbers of periodically firing neurons and more on low degrees of cooperativity achieved by order parameters influencing very large numbers of neurons.

This evidence supports the hypothesis of long range synchronizing forces. Such forces maybe internal to the brain or they may be external entraining forces. An example of the latter is a metric perturbation, which is studied in this work, as applied to synchronizing systems of axons.

Synchronization, a type of coupling, can also be seen as a way of extraction of information about a system. One puts a known smaller system in synchrony with a larger or much larger system already operating as per a synchronization law and thereby, by observing the smaller system, gains information such as the phase and frequency of the synchronization. In this light, one can even view any physical experiment as follows. The experimenter prepares the experimental apparatus in such a way that it synchronizes with a larger system exhibiting a particular law or phenomenon. Thereby the apparatus starts to yield information about that law or phenomenon.

For example, a child prepares a top and "throws" it and it begins spinning until it precesses and falls to a stop. Here the child can extract some information about the nature of gravitational force by observation of the spinning and precession. This was done by putting the top in synchrony with the larger body, the Earth and its gravitational aspect.

The study of axon tracts under coupling conditions [2, 3, 4] has led to many insights into the relation between geometry and coupling. Using this study and the models generated in [5], we were recently able to show the interaction between metric perturbations (which involve the geometry of spacetime) and tracts and extend that to propose a mechanism for brain quantum computation [6, 7]. Information theoretic studies in this domain are also quite fascinating [8, 9, 10].

An axon is the extended or elongated part of a nerve cell or neuron. It extends from the axon hillock which is adjacent to the cell body, to the axonal terminal bouton. These terminal boutons are usually depicted as lone interaction sites with the next neuron, but some cells also have axonal branching which leads to multiple such boutons. The design of the nerve cell with a spherical cell body and a single projecting axon brings to mind the thought that the brain is inherently aware of spatial separation. As spatial separation is emphasized, the question becomes one of how temporal separation is represented.

Suppose two signals impinge on the soma separated by t_1 seconds. If we were to create a short to the terminal bouton, the outputs would also be separated by t_1 seconds. However, there is intervening processing and these signals are each encoded first. Even if encoded temporally, the t_1 seconds separation is not maintained because the cell body processes them as post-synaptic potentials which must gather in sufficient quantity to trigger an action potential, and this could take more than t_1 seconds. Thus, many temporal transformations occur between the input and the output. This implies that initial temporal separations are modified in general. This change in temporal representation, or Δt, is captured as a change in the entropy of the output with respect to the input. This is the information processing taking place in a nerve cell. Were the nerve cell of relativistic dimensions, a change in temporal representation would be concomitant with a change in spatial representation as well. Thus, the heuristic we are led to is that a nerve cell's spatio-temporo-entropic transformation is governed by a relation such as $(\Delta x, \Delta t) = f(\Delta I)$. A nerve cell is a stand-in for the general information processing machine. An axon tract is also an information processing machine, albeit a composite one, so this relation governs each axon tract as well.

2 Methods and Results

Consider an axon tract stretching from the axis coordinate $z = 0$ to $z = L$. The tract contains axons, four in number, at various mutual distances a, b, c and d, and various inclinations θ_i with respect to the z-axis. Such a tract, with one less axon for clarity, is shown illustrated in Figure 1. We begin our investigation with a question as to whether, instead of a universal coordinate, an 'averaged' coordinate can be used for a joint view of all the axons' voltage profiles. To address this question, we present the first theorem, preceded by necessary definitions.

Definition 1. Constant Integer Factor.

This is defined in the present paper as an integer or a ratio of integers.

Definition 2. Universal Model.

This refers to the derivation presented in [2], Equations (1)-(10), when the z-coordinate is constructed as the projection of the s-coordinate onto the tract axis, for each axon. It is universal in the sense that all the projections are along the same direction.

Definition 3. Typical Model.

This refers to the derivation presented in [2], Equations (1)-(10), when the z-coordinate is constructed as an average of the projections of the s-coordinates onto the tract's axis. It is typical in the sense that just the averaged value is used.

Definition 4. Equivalent.

Two models are said to be equivalent if the governing relation between derivatives in both is identical.

Theorem 5. *Within a constant integer factor, the universal model, specified by a universal coordinate, is equivalent to the typical model, specified by an averaged coordinate.*

Proof. The universal model is specified by a universal coordinate z_U, such that

$$z_U = s_i \cos\theta_i \tag{1}$$

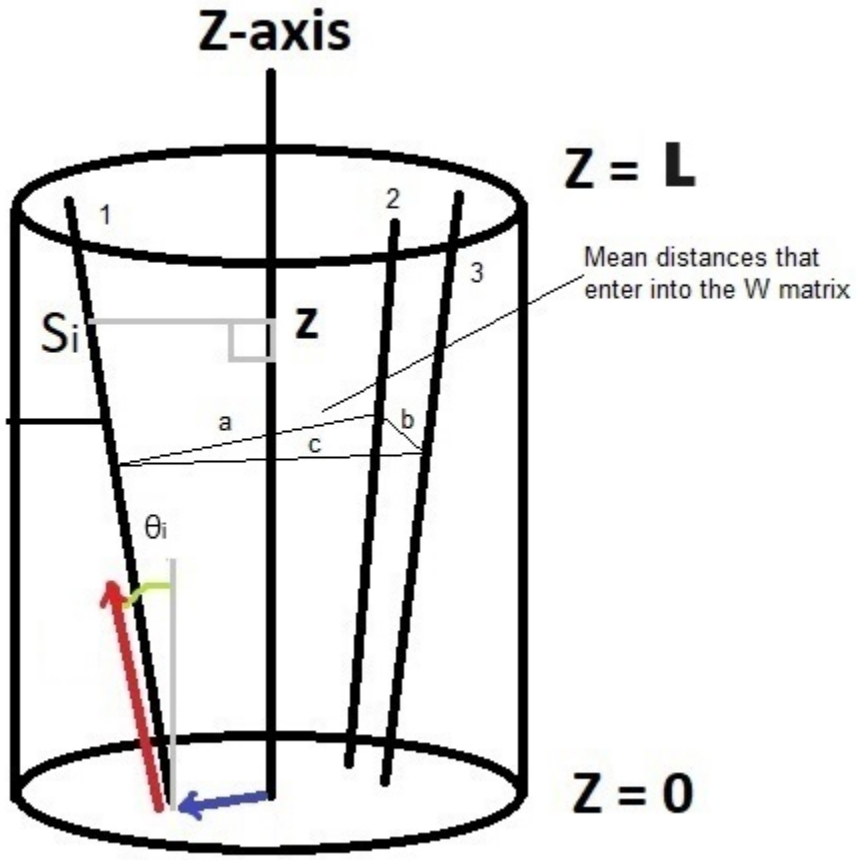

Figure 1: Axon tract showing three axons inclined with respect to the tract axis. Figure adapted from [2].

where s_i, for $i \in \{1, \ldots, N\}$, are the coordinates along the i-th axon and N is the total number of axons in the tract under consideration. The typical model on the other hand, is specified by an averaged coordinate thus:

$$z_A = \frac{s_1 \cos\theta_1 + s_2 \cos\theta_2 + s_3 \cos\theta_3 + s_4 \cos\theta_4}{4}. \tag{2}$$

Instead of assuming that we are looking at equal projections from each axon's presently observed position, we are looking at possibly distinct projections from the various axons, and average them out before employing them in the derivation of the tract propagation equation. That is,

$$z_A = \frac{\sum_{i=1}^{N} s_i \cos\theta_i}{N}. \tag{3}$$

This yields, upon differentiation with respect to the along-axon coordinates,

$$\frac{\partial[\cdot]}{\partial s_i} = \frac{\cos\theta_i}{N} \frac{\partial[\cdot]}{\partial z_A}. \tag{4}$$

This implies that the universal model is, within a constant tract-size (integer) factor N, nothing but the averaged model, cf. [2]. □

It leads us to our next theorem which relates synchronization in the two models to one another.

Theorem 6. *Synchronization in the typical model implies normal synchronization, within certain conditions.*

Proof. Synchronization in the typical model means that post-synchronization we will have,

$$\frac{\partial V_1}{\partial z_A} = \ldots = \frac{\partial V_N}{\partial z_A} \tag{5}$$

and pre-synchronization these partial derivatives may differ. This is a condition of phase synchronization: as the coordinate is increased, the transmembrane voltage rises by an equal extent in each axon.

For normal synchronization, where we look at individual axons and find that their actual rates of change of transmembrane voltage V_i with respect to local coordinate s_i are identical to each other, the quantity to look at is $\frac{\partial V_i}{\partial s_i}$. We may approximate the partial derivative with ratios of delta amounts. Let δ_1 represent a small change in s_1 and so on for δ_2 etcetera. With z_A as defined in Equation (3), we have

$$\triangle z_A = \frac{1}{N} \sum_{i=1}^{N} \delta_i \cos\theta_i. \tag{6}$$

Suppose we are able to see z_A-synchronization. Then Equation (5) yields that $\triangle V_i$ are identical. So we may write

$$\frac{\triangle V_1}{\delta_1 \cos\theta_1} = \frac{\triangle V_2}{\delta_2 \cos\theta_2} = \ldots. \tag{7}$$

Next, let

$$\|\delta_i \cos\theta_i - \triangle z_A\| = \varphi_i. \tag{8}$$

Suppose $\varphi_i \ll 1$ for all i. Then

$$\frac{\triangle V_N}{\delta_N \cos\theta_N} \approx \frac{\triangle V_N}{\triangle z_A} \approx \frac{\triangle V_N}{\delta_1 \cos\theta_1} = \frac{\triangle V_1}{\delta_1 \cos\theta_1}. \tag{9}$$

That is,

$$\frac{\triangle V_i}{\delta_i} \approx \frac{\cos\theta_i}{\cos\theta_1} \cdot \frac{\triangle V_1}{\delta_1} \tag{10}$$

Thus when $\varphi_i \ll 1$ for all i is satisfied and the axons are equally inclined to the tract axis, typical synchronization implies normal or actual synchronization in the phase sense. □

S. No.	Inclinations (all four axons; in degrees)	Gap between CV1 and CV2
1.	0	4.8913e6
2.	10	1.1364e5
3.	20	n/a on run 1; 3.4453e4 on run 2
4.	30	2.4272e4
5.	40	-2.4742e6
6.	50	-9.3556e4
7.	60	n/a on three runs

Table 1: Results of Simulations. CVi = instantaneous conduction velocity on axon i, at a fixed, common node.

3 Discussion and Conclusions

This paper clarifies the results presented in [2]. The first theorem, Theorem 5, tells us how to think about the universal coordinate (Section 2 in [2]) and the second theorem, Theorem 6, tells us that when we see synchronization as per the universal coordinate of that paper, it implies actual synchronization, though only for certain geometries.

We validated these results using the programs developed in [2] and [5]. The results of those simulations are shown in Table 1. The theory holds for small angles. Beyond 45 degrees, the projection onto the $z = 0$ plane is larger for each axon, than on the universal coordinate. Thus the discrepancy for those values. Small amounts of random noise were also injected at each node.

Synchronization is a temporal transformation since the input temporal specification is different from that at the output. As discussed in the introductory section, the relation $(\Delta x, \Delta t) = f(\Delta I)$ tells us that a temporal transformation would be linked to an entropic transformation. Thus we are urged to look for the entropic transformation concomitant with synchronization and readily find it in the fact that, if time coded, the information pattern pre-synchronization differs from the information pattern post-synchronization. A detailed mathematical study of the two theorems presented herein might allow us to deduce the form of the function f in terms of tract parameters so that we may be able to view the tract as a transducer between entropy and time. This view, extended to the whole nervous system, has ramifications for our understanding of man's perception of space and time.

Some of the limitations of this work include the fact that we did not consider additional techniques of forming the z coordinate which would then be compared with the two models considered herein. Another limitation, which can be addressed in future work, is that we didn't answer the question of whether there is any way to remove the restriction on geometries which allow for the equivalence considered herein.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and materials

All data generated or analysed during this study are included in this published article

Competing interests

Not applicable.

Funding

Not applicable.

Authors' contributions

AC conceived the study, developed the theory, obtained the results and wrote the manuscript.

Acknowledgments

Not applicable.

References

[1] Walter J Freeman. Characteristics of the synchronization of brain activity imposed by finite conduction velocities of axons. *International Journal of Bifurcation and Chaos*, 10(10):2307–2322, 2000.

[2] Aman Chawla, Salvatore Morgera, and Arthur Snider. On axon interaction and its role in neurological networks. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 2019.

[3] Aman Chawla. *On Axon-Axon Interaction via Currents and Fields.* PhD thesis, University of South Florida, 2017.

[4] Aman Chawla and Salvatore Domenic Morgera. On the geometry of interacting axon tracts in relation to their functional properties. *bioRxiv*, 2022.

[5] Aman Chawla, Salvatore D. Morgera, and Arthur D. Snider. Fields, geometry, and their impact on axon interaction. *Journal of Applied Mathematics and Physics*, 9(4):751–778, 2021.

[6] Aman Chawla. A gedanken experiment on the brain as a quantum computer. *ResearchGate Preprint*, 2022.

[7] Aman Chawla and Salvatore Domenic Morgera. Mechanism of universal quantum computation in the brain. *Journal of Applied Mathematics and Physics*, 12(2):468–474, 2024.

[8] Aman Chawla and Salvatore D Morgera. Ephaptic synchronization as a mechanism for selective amplification of stimuli. *BMC Neuroscience*, 15(1):1–1, 2014.

[9] Aman Chawla and S. D. Morgera. Ephaptic coupling in axon tracts under time and rate coding of information: a computer study. *Bernstein Conference*, 2020.

[10] Aman Chawla and Salvatore Domenic Morgera. On the error exponent of axon-coupled channels: Computer explorations. In *Bernstein Conference 2022*, pages PIII–19, 2022.