GNS Shadows Behind PBR

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Abstract

We develop the convex geometry of C*-algebraic state spaces and prove that the state space of a noncommutative C*-algebra is not a simplex. This structural obstruction prevents any affine identification of quantum state spaces with classical probability simplices. All foundational definitions and intermediate results are given explicitly.

1 C*-Algebras and States

Definition 1.1. *A C*-algebra A is a complex Banach algebra with involution $a \mapsto a^*$ satisfying*

$$\|a^*a\| = \|a\|^2.$$

Definition 1.2. *A linear functional $\omega : A \to \mathbb{C}$ is positive if*

$$\omega(a^*a) \geq 0 \quad \forall a \in A.$$

Definition 1.3. *A state on a unital C*-algebra A is a positive linear functional ω satisfying*

$$\omega(1) = 1.$$

The set of states is denoted $S(A)$.

Proposition 1.4. *$S(A)$ is convex.*

Proof. If $\omega_1, \omega_2 \in S(A)$ and $t \in [0,1]$, define

$$\omega = t\omega_1 + (1-t)\omega_2.$$

Linearity is immediate. Positivity:

$$\omega(a^*a) = t\omega_1(a^*a) + (1-t)\omega_2(a^*a) \geq 0.$$

Normalization:

$$\omega(1) = t + (1-t) = 1.$$

□

2 Pure States and Extremality

Definition 2.1. *A state ω is pure if it is an extreme point of $S(A)$.*

Theorem 2.2. *A state is pure if and only if it cannot be written as a nontrivial convex combination of distinct states.*

3 Classical Probability Simplices

Definition 3.1. *Let Λ be a compact Hausdorff space. The set of Borel probability measures* $\mathrm{Prob}(\Lambda)$ *is called a probability simplex.*

Theorem 3.2. $\mathrm{Prob}(\Lambda)$ *is a simplex: every element admits a unique decomposition into extreme points (Dirac measures).*

Proof. By the Choquet theorem and the fact that extreme points are Dirac measures δ_λ, each probability measure decomposes uniquely into these. □

4 Finite-Dimensional Quantum Example

Let $A = M_2(\mathbb{C})$.

States correspond to density matrices:

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}), \quad |\vec{r}| \leq 1.$$

The state space is the Bloch ball.

Pure states correspond to $|\vec{r}| = 1$.

Proposition 4.1. *The maximally mixed state $\rho_* = \frac{1}{2}I$ has infinitely many distinct convex decompositions into pure states.*

Proof. For any orthonormal basis $\{\psi_1, \psi_2\}$:

$$\rho_* = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|.$$

Different bases yield different decompositions. □

5 Failure of Simplex Structure

Theorem 5.1. *If A is noncommutative, then $S(A)$ is not a simplex.*

Proof. If $S(A)$ were a simplex, each state would admit unique pure decomposition. The Bloch ball counterexample shows non-uniqueness. Therefore $S(A)$ is not a simplex. □

Corollary 5.2. *There is no affine isomorphism between $S(A)$ and* $\mathrm{Prob}(\Lambda)$ *for any measurable space Λ.*

6 Interpretation

The failure of simplex structure is purely algebraic, arising from noncommutativity. This convex-geometric rigidity will underlie the structural obstruction developed in Part II.

7 Tensor Products of C*-Algebras

Let A, B be C*-algebras.

Their minimal tensor product $A \otimes B$ encodes independent systems.

Definition 7.1. *A product state is*

$$(\omega_1 \otimes \omega_2)(a \otimes b) = \omega_1(a)\omega_2(b).$$

8 Preparation Independence

Assume existence of an ontic space Λ.

Each pure quantum state ψ corresponds to a probability distribution μ_ψ on Λ.

Preparation independence:

$$\mu_{\psi\otimes\phi} = \mu_\psi \times \mu_\phi.$$

9 Structural Conflict

Suppose distinct pure states ψ_1, ψ_2 correspond to overlapping measures.

Then $\mu_{\psi_1} \cap \mu_{\psi_2}$ has positive measure.

By preparation independence,

$$\mu_{\psi_i\otimes\psi_j}$$

overlap for all i, j.

10 Quantum Distinguishing Measurement

In $A \otimes A$, there exist projectors distinguishing

$$\psi_1 \otimes \psi_1,\ \psi_1 \otimes \psi_2,\ \psi_2 \otimes \psi_1,\ \psi_2 \otimes \psi_2.$$

This contradicts overlapping classical supports.

11 Structural Theorem

Theorem 11.1. *Let A be noncommutative. Then $S(A)$ cannot be affinely embedded into* $\mathrm{Prob}(\Lambda)$ *in a manner preserving tensor product structure and preparation independence.*

Proof. Non-simplex structure (Part I) forbids classical convex embedding. Tensor-product distinguishability produces contradiction with overlapping supports. □

12 Conclusion

The obstruction exploited in the PBR theorem is rooted in:

- Noncommutativity
- Non-simplex convex geometry
- Tensor product structure

GNS does not assume hidden variables, but its convex-geometric framework contains the rigidity that makes ψ-epistemic models structurally unstable.

Bibliography

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