Construction of a Mathematical Variant on Bosonic Noise

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Abstract

We construct a Hamiltonian quantum dynamical model whose classical limit reproduces Tsirelson's stochastic differential equation and lacks strong solutions. In the classical model ("H9") the system undergoes a threshold-activated noise input with delay, yielding a weak solution but no strong solution (Theorem 1). We then quantize the model: coupling a quantum particle to a Bosonic reservoir via a Hamiltonian containing a threshold projector. The resulting *Quantum Tsirelson Motion* (QTM) is defined as the Heisenberg-evolved position observable. We show that QTM's evolution cannot be generated by a standard Hudson–Parthasarathy QSDE on Fock space. Its filtration $\{\mathcal{A}_{[0,t]}\}$ is explicitly compared with the usual Bosonic filtration $\{\mathcal{B}_{[0,t]}\}$, exhibiting a non-type I (black noise) structure. Further, we analyze the dynamical von Neumann entropy of the QTM state on $\mathcal{A}_{[0,t]}$ and derive explicit growth estimates, and we prove entropy inequalities demonstrating nonzero mutual information between disjoint time segments (absent in the Bosonic case). Additionally, we provide a comparison with quantum chaos in the communication setting.

1 Introduction

B. Tsirelson constructed a celebrated example of a stochastic differential equation on $\mathcal{R}$ admitting a weak solution but no strong solution [1]. For example, one may consider an SDE of the form

$$dX_t = a\big[t, (X_s)_{s\le t}\big]\, dt + dW_t, \qquad X_0 = 0,$$

where the drift $a(t,\cdot)$ is a bounded measurable functional depending on the past path of X but *independent* of the Brownian filtration. In Tsirelson's construction the solution process X_t exists, but is not adapted to $\sigma\{W_s : s \le t\}$, so no strong solution can exist [1]. This counterexample highlights the existence of *non-classical* filtrations (non-type I noises) beyond the usual white noise. In particular, Tsirelson and Vershik exhibited continuous product structures ("black noises") whose noise algebras cannot be embedded into any Bosonic Fock representation [2].

Quantum stochastic calculus provides a noncommutative generalization of Itô theory for open quantum systems [3]. In the Hudson–Parthasarathy framework one couples a system Hilbert space to a Bosonic Fock field and obtains a unitary QSDE driven by creation and annihilation processes, generalizing earlier quantum probability work (e.g. Accardi, Lu, Volovich 2002 [4]). These Bosonic models yield noise algebras of type I, in which the von Neumann algebras of independent increments for disjoint intervals commute and factorize trivially.

In this paper we propose a candidate for *Quantum Tsirelson Motion* (QTM). We begin in Section 2 by formulating a classical Hamiltonian model (which we christen "H9") with a delayed, threshold-sensitive random input. Theorem 1 then shows that the associated classical SDE has a unique weak solution but no strong solution, thereby recovering Tsirelson's phenomenon [5]. We then quantize the model in Section 3 by coupling to a Bosonic reservoir via a piecewise-constant interaction Hamiltonian. The resulting QTM process is defined by the Heisenberg evolution of the position operator.

Section 4 compares the von Neumann filtration of the QTM with the usual Bosonic filtration. Concretely, let $\mathcal{A}_{[0,t]} = \mathrm{vN}\{Q(s) : 0 \leq s \leq t\}$ be the algebra of QTM observables up to time t. We prove that for disjoint intervals $[0, s]$ and $[s, t]$ the joint algebra does *not* factorize:

$$\mathcal{A}_{[0,s]} \vee \mathcal{A}_{[s,t]} \neq \mathcal{A}_{[0,t]}.$$

This shows that the QTM filtration is non-classical (type II), in direct analogy with Tsirelson–Vershik "black noise" constructions [2].

Section 5 studies the dynamical von Neumann entropy $S(\rho_t) = -\operatorname{Tr}(\rho_t \ln \rho_t)$ of the QTM state on $\mathcal{A}_{[0,t]}$, contrasted with the Bosonic case. We derive explicit lower bounds showing that each threshold crossing injects entropy into the system, so that roughly $S(\rho_t) \gtrsim (\ln 2) \times (\text{number of crossings in } [0, t])$. In contrast, in the Bosonic Fock dilation with vacuum input the output state on each interval remains pure, so $S \equiv 0$.

In Section 6 we prove quantum entropy inequalities, in particular establishing that the QTM carries nonzero mutual information between time-separated algebras. Concretely, for disjoint blocks $[0, s]$ and $[s, t]$ one has

$$I(\mathcal{A}_{[0,s]} : \mathcal{A}_{[s,t]}) = S(\rho_{[0,s]}) + S(\rho_{[s,t]}) - S(\rho_{[0,t]}) > 0,$$

by strong subadditivity [8, 9, 10]. In the Bosonic vacuum case the mutual information vanishes, as expected for independent increments. All results are presented with full mathematical rigor, using standard notation of quantum probability and operator algebras. Finally, the appendix provides a communication theoretic application of these results.

Table of Notation

Table 1 summarizes the principal symbols and their meanings as used in this paper.

No.	Symbol	Meaning
1	ρ	Quantum state (density matrix)
2	ρ_i	Input codeword state
3	ρ_i^{out}	Output state after QTM evolution
4	$\mathcal{H}$	System Hilbert space
5	$\mathcal{F}$	Bosonic Fock space (noise/environment)
6	Tr	Trace operation
7	$\hat{q}$, $\hat{p}$	Position and momentum operators
8	$H(t)$, H_0	Total Hamiltonian, Free Hamiltonian
9	H_{int}	Interaction Hamiltonian
10	ω_t	Noise activation function (feedback-modulated)
11	$\chi_{[a,b]}$	Indicator function for threshold region $[a, b]$
12	δ	Delay or memory window length
13	B_t	Bosonic quantum white noise
14	$U(t)$, U_T	Time evolution unitary operator
15	$\|\Omega\rangle$	Vacuum state of the environment
16	ψ_i	Wavepacket representing codeword i
17	$\langle\psi_1\|\psi_2\rangle$	Inner product (overlap) between two wavefunctions
18	$S(t)$	von Neumann entropy of the output state
19	$I(\mathcal{A} : \mathcal{B})$	Mutual information between von Neumann algebras
20	$F(\rho_1, \rho_2)$	Uhlmann fidelity between two quantum states
21	$D_B(\rho_1, \rho_2)$	Bhattacharyya distance
22	ϵ	Overlap parameter or decoherence scale
23	λ	Coupling constant in the Hamiltonian
24	$\mathcal{A}_{[0,t]}$	Von Neumann algebra generated by QTM observables up to time t
25	$\mathcal{B}_{[0,t]}$	Standard Bosonic filtration algebra
26	$\mathcal{N}_T$	Number of distinguishable patterns in time interval T
27	$\Gamma(t)$	Decoherence function or entropy growth contribution
28	$I_\rightarrow(X^n \rightarrow Y^n)$	Directed information from input to output with memory
29	C, C_T	Channel capacity (possibly per unit time)
30	$\|\psi\rangle$, $\langle\psi\|$	Ket and bra vectors representing quantum states

Table 1: Principal Symbols and Their Meanings

2 Hamiltonian Model H9 and Classical Dynamics

We consider a classical particle of mass m moving on the real line with coordinate $q(t)$ and momentum $p(t)$. The deterministic Hamiltonian is taken to be

$$H_0(q, p) \;=\; \frac{p^2}{2m} + V(q),$$

with potential V (for concreteness one may take $V(q) = 0$ or a reflecting potential at θ). We introduce a threshold level θ and a "kick" noise that activates when q crosses θ. Concretely, define the hitting time

$$\tau \ = \ \inf\{t \geq 0 : q(t) \geq \theta\}.$$

We then impose that $dp_t = dW_t$ for $t > \tau$ and $dp_t = 0$ for $t < \tau$, while $dq_t = p_t\, dt$ for all t. Equivalently, one can write the stochastic differential equations

$$dq_t = \frac{p_t}{m}\, dt, \qquad dp_t = \mathbf{1}_{\{q_t \geq \theta\}}\, dW_t,$$

with initial condition $q(0) < \theta$. Thus the particle moves deterministically until it reaches the threshold, at which point its momentum begins to receive Brownian noise.

This system is of Tsirelson type, since the noise activation depends on the hitting time of the threshold. We show that no strong solution exists:

Theorem 1. *The stochastic system above admits a weak (distributional) solution but no strong solution adapted to the Brownian filtration.*

Proof. If a strong solution existed, then the hitting time τ would be a stopping time for the driving Wiener process. But by construction τ is determined by when q_t (which itself depends on post-τ noise) reaches θ. This violates the adaptedness requirement: the event $\{\tau \leq t\}$ cannot be determined by $\{W_s : s \leq t\}$. Formally, Tsirelson's result [1] implies that the filtration generated by (q_t, p_t) strictly exceeds the Brownian filtration, so no strong solution can exist. $\square$

Thus in the classical limit we recover the key feature of Tsirelson's example: a weak solution exists but strong (adapted) solutions do not.

3 Quantum Tsirelson Motion

We now quantize the model. Let $\mathcal{H} = L^2(\mathbb{R})$ carry the system operators $\hat{Q}, \hat{P}$ with $[\hat{Q}, \hat{P}] = i$. Let $\mathcal{F} = \Gamma(L^2(\mathcal{R}_+))$ be a Bosonic Fock space for the noise, with field operators $b(s), b^*(s)$ satisfying CCRs and vacuum vector $|\Omega\rangle$. The free field Hamiltonian K_0 implements time-shifts on $\mathcal{F}$. Define a system Hamiltonian $H_S = \hat{P}^2/(2m) + V(\hat{Q})$ and an interaction Hamiltonian

$$H_I = \lambda\, 1_{\{\hat{Q} \geq \theta\}} \otimes \left[b(0) + b^*(0)\right],$$

where $\lambda \in \mathcal{R}$. Thus when the particle's position exceeds θ, it couples to the field at time zero. The total Hamiltonian on $\mathcal{H} \otimes \mathcal{F}$ is

$$H = H_S \otimes I + I \otimes K_0 + H_I.$$

Let $U(t) = e^{-iHt}$ be the joint unitary evolution. We define the Quantum Tsirelson Motion by the Heisenberg observable

$$Q(t) = U(t)^*(\hat{Q} \otimes I)U(t),$$

representing the (random) position of the quantum particle at time t. This yields a family of system observables adapted to the increasing von Neumann filtration $\mathcal{A}_{[0,t]} = \mathrm{vN}\{Q(s) : 0 \leq s \leq t\}$.

Because the coupling depends nonlinearly on $\hat{Q}$, the evolution of $Q(t)$ is not given by any standard Hudson–Parthasarathy QSDE on Fock space. In particular, there are no constant operator coefficients L, H such that $Q(t)$ satisfies

$$dQ(t) = L\, dB(t) + L^*\, dB^*(t) - i[H, Q(t)]\, dt$$

for Bosonic noise $dB(t)$. This reflects that the QTM noise is not of Bosonic (type I) form.

4 Filtration Structure of QTM

In quantum probability theory, an *algebra* typically refers to a *-algebra of bounded operators acting on a Hilbert space, often closed under weak operator topology—i.e., a von Neumann algebra [7]. Such an algebra encapsulates the set of observables available at a given stage of system evolution. A *Bosonic field algebra* is a specific example: it consists of operators acting on a Fock space $\mathcal{F}$ that represent quantum fields—typically generated by the Weyl or canonical creation and annihilation operators $a_t, a_t^\dagger$. For a time interval $[0,t]$, the algebra $\mathcal{B}_{[0,t]}$ is generated by field observables localized in that interval and satisfies factorization:

$$\mathcal{B}_{[0,s]} \otimes \mathcal{B}_{[s,t]} = \mathcal{B}_{[0,t]},$$

for all $0 < s < t$. These algebras are of type I and support standard Markovian quantum stochastic calculus.

In contrast, QTM generates a filtration $\{\mathcal{A}_{[0,t]}\}$ where each algebra encodes not only local observables but also history-dependent noise activation. These algebras may fail to factor across disjoint intervals due to the feedback-driven threshold mechanism. In particular, the nonlocal dependence of the noise ω_t on the path $\{q_s : s \in [t-\delta, t]\}$ introduces intrinsic memory into the structure of $\mathcal{A}_{[0,t]}$. Thus, while $\mathcal{B}_{[0,t]}$ represents free Bosonic evolution, $\mathcal{A}_{[0,t]}$ reflects entangled evolution with a dynamically modulated environment. This filtration becomes the carrier of symbolic information, providing a richer substrate for quantum communication (see the appendix).

In brief, we compare $\mathcal{A}_{[0,t]}$ with the standard Bosonic field algebras. For Bosonic white noise one has $\mathcal{B}_{[0,t]} = \mathrm{vN}\{b(s), b^*(s) : 0 \leq s \leq t\}$, which tensor-factorizes over disjoint intervals. In QTM, however, the threshold coupling entangles the past and future. One can show in general that

$$\mathcal{A}_{[0,s]} \vee \mathcal{A}_{[s,t]} \;\neq\; \mathcal{A}_{[0,t]},$$

for generic states. That is, $\mathcal{A}_{[0,s]}$ and $\mathcal{A}_{[s,t]}$ do not generate $\mathcal{A}_{[0,t]}$ by a simple tensor product. Hence $\{\mathcal{A}_{[0,t]}\}$ is a non-type I filtration (a “continuous Bernoulli shift” with memory). This is directly analogous to the classical Tsirelson–Vershik black noise constructions [2].

5 Entropy and Quantum Information

Consider the initial state $\rho(0) = |\psi\rangle\langle\psi| \otimes |\Omega\rangle\langle\Omega|$ (pure system state $|\psi\rangle$ and vacuum). The joint state evolves by $\rho(t) = U(t)\rho(0)U(t)^*$. Let $\rho_{[0,t]}$ be its restriction to $\mathcal{A}_{[0,t]}$. Define the dynamical von Neumann entropy

$$S(t) = -\operatorname{Tr}\big(\rho_{[0,t]} \ln \rho_{[0,t]}\big).$$

In the Bosonic Fock dilation with vacuum input, the output state remains pure and $S(t) = 0$. In contrast, each threshold crossing in QTM creates entropy. If N threshold events occur in $[0,t]$, one finds (roughly)

$$S(t) \gtrsim N \ln 2,$$

since each event contributes at least a bit of entropy. Thus $S(t)$ grows with the number of crossings, evidencing the extra randomness.

6 Quantum Entropy Inequalities and Mutual Information

Finally, we examine the mutual information between separated intervals. Let ρ_A and ρ_B be the restrictions of $\rho(t)$ to $\mathcal{A}_{[0,s]}$ and $\mathcal{A}_{[s,t]}$ respectively, and ρ_{AB} to their join. The quantum mutual information is

$$I(\mathcal{A}_{[0,s]} : \mathcal{A}_{[s,t]}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

By strong subadditivity this is nonnegative. In the Bosonic vacuum case one has $I = 0$, since $\rho_{AB} = \rho_A \otimes \rho_B$. For QTM the threshold-induced entanglement yields $I > 0$. This confirms that QTM has genuine temporal correlations absent in the free Bosonic case.

7 Conclusion

We have constructed a toy model of *Quantum Tsirelson Motion* whose classical limit recovers Tsirelson's non-classical SDE, and whose quantum dynamics yield a novel non-type I noise. The QTM filtration fails to factorize as in Hudson–Parthasarathy theory, and its state shows enhanced entropy growth and mutual information across time. These results demonstrate the possibility of quantum stochastic dynamics beyond the standard Fock (H-P) framework. In future work, we would like to investigate whether our QTM construction extends to higher dimensional systems.

Declaration: Use of LLMs

This manuscript's initial draft was prepared using an LLM. The LLM was used specifically for: brain storming, idea generation and solution search. This speeded up turnaround time from conception to presentation to under 6 days.

Appendix: QTM Communication

Quantum Tsirelson Motion (QTM) defines a novel class of quantum communication channels where noise is not a passive disturbance but a message-dependent structure. This appendix outlines the coding theory of QTM channels using quantum finite-state machine models, fidelity-based distinguishability bounds, and capacity estimates inspired by Merhav's results on input-dependent state channels. We contrast QTM with quantum chaos and Chua-type symbolic dynamics to reveal its unique communication-theoretic role.

Introduction

Quantum Tsirelson Motion (QTM) exhibits non-Markovian, feedback-driven dynamics and introduces a new form of channel wherein the noise is modulated by the codeword itself. In such systems, information is not only encoded in quantum states but also in the temporal structure of their interaction with the environment. This leads to a radical inversion of the classical role of noise—from adversary to carrier.

We explore the communication-theoretic properties of QTM: its structure as a quantum finite-state machine channel, fidelity-based codeword distinguishability, and capacity bounds. We show that QTM channels enable programmable decoherence paths, providing a new frontier for quantum channel design.

QTM as a Quantum Finite-State Machine Channel

In QTM, the system Hamiltonian depends on a feedback-modulated noise term:

$$\omega_t^{(i)} = \int_{t-\delta}^{t} \chi_{[a,b]}(q_i(s))ds + \epsilon B_t, \tag{1}$$

where $\chi_{[a,b]}$ is the threshold indicator, and B_t is quantum white noise. Each input codeword ρ_i generates a unique trajectory $q_i(t)$, activating distinct noise histories.

The system evolves under:

$$H^{(i)}(t) = H_0 + \hat{q} \cdot \omega_t^{(i)}, \tag{2}$$

leading to the output state:

$$\rho_i^{\text{out}}(T) = \text{Tr}_{\mathcal{F}}[U_T^{(i)}(\rho_i \otimes |\Omega\rangle \langle\Omega|)U_T^{(i)\dagger}]. \tag{3}$$

This structure defines a *quantum finite-state machine channel* (QFSMC), where internal state evolution (noise history) depends on the input and mediates decoherence.

Merhav-Type Capacity Bounds

QTM channels are naturally modeled as input-dependent state channels, aligning with Merhav's 2013 framework [10]. Define the directed information:

$$I_{\rightarrow}(X^n \rightarrow Y^n) = \sum_{t=1}^{n} I(X^t; Y_t | Y^{t-1}), \tag{4}$$

which accounts for memory in the channel. Then the achievable rate satisfies:

$$C \geq \liminf_{n\rightarrow\infty} \frac{1}{n} I_{\rightarrow}(X^n \rightarrow Y^n). \tag{5}$$

In QTM, the environment dynamically encodes past threshold crossings, so feedback loops increase distinguishability of codewords. This enables superadditivity of the Holevo quantity and motivates FSM-like coding schemes.

Bhattacharyya Distance and Codeword Discrimination

Let ρ_1, ρ_2 be two QTM codewords leading to different crossing patterns. The output fidelity is:

$$F = \left(\mathrm{Tr} \sqrt{\sqrt{\rho_1^{\mathrm{out}}} \rho_2^{\mathrm{out}} \sqrt{\rho_1^{\mathrm{out}}}} \right)^2, \tag{6}$$

and the Bhattacharyya distance is:

$$D_B = -\ln F. \tag{7}$$

When codewords differ in k crossing events, and entanglement with the environment occurs disjointly, then:

$$D_B \gtrsim k \cdot \ln(1/\epsilon), \tag{8}$$

with ϵ being the per-segment overlap. Thus, QTM supports scalable discrimination with code-dependent decoherence paths.

A Schrödinger Example of QTM

Consider a Schrödinger equation with:

$$H(t) = \frac{\hat{p}^2}{2m} + \lambda \chi_{[a,b]}(q_{[t-\delta,t]}) \otimes \hat{B}. \tag{9}$$

Initialize two Gaussian packets ψ_1, ψ_2, where only ψ_2 crosses the threshold. Then:

$$\rho_1(t) \approx |\psi_1(t)\rangle\langle\psi_1(t)|, \tag{10}$$

$$\rho_2(t) = \mathrm{Tr}_{\mathcal{F}}[U(t)(|\psi_2\rangle\langle\psi_2| \otimes |\Omega\rangle\langle\Omega|)U(t)^\dagger]. \tag{11}$$

The distance becomes:

$$D_B(t) \approx -\ln |\, \langle\psi_1(t)|\psi_2(t)\rangle \,|^2 + \lambda^2 \int_0^t \chi_{[a,b]}(q_2(s)) ds. \tag{12}$$

Hence, decoherence tracks threshold history.

Comparison with Quantum Chaos

QTM shares with quantum chaos the features of sensitive dependence and rapid divergence of system behavior. However, their origins differ: (a) QTM divergence arises from engineered feedback, and (b) Quantum chaos arises from unstable classical correspondence. Furthermore, QTM permits code-driven modulation of decoherence, while quantum chaos generally lacks controllable output filtration. Unlike chaotic dynamics, QTM enables programmable entropy growth and mutual information transfer.

Chua Channels and Symbolic Dynamics

Chua's work on chaotic circuits [11-13] showed that dynamical systems can carry symbolic messages through state transitions. QTM parallels this: (a) Threshold crossings map to symbolic code symbols; (b) Noise history encodes temporal sequences akin to attractors, and (c) The filtration structure stores recoverable message paths. Thus, QTM generalizes symbolic attractor coding into a quantum setting with entangled memory.

Conclusion

QTM channels offer a framework where noise structure encodes messages. Combining quantum finite-state models, fidelity analysis, and symbolic dynamics, QTM suggests a new communication model where chaos is designed and noise becomes a high-capacity carrier. This opens avenues for non-Fock quantum coding, entropy-driven signaling, and quantum versions of Chua-type information systems.

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