Communication-on-the-fly With High Quality Bits - Part I: Motivation

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Abstract

In this paper (part I), the authors sketch a high-level overview of the motivation for anytime information theory.

1 Preliminaries

1.1 Information Theory

The reader unfamiliar with information theory can read Prof. Thomas Cover's book [1] for the various types of channels including the erasure channel, properties of channel capacity, and the noisy channel coding theorem of Claude Shannon.

1.2 Control Theory

We follow the development in [2]. Let a Finite Dimensional Linear Time Invariant (FDLTI) dynamical system be described by the following linear constant-coefficient differential equations:

$$\dot{x} = Ax + Bu \tag{1}$$

where

$$x(t_0) = x_0 \tag{2}$$

and

$$y = Cx + Du \tag{3}$$

where $x(t)$ in R_n is called the system state, $x(t_0)$ is called the initial condition of the system, $u(t)$ in R_m is called the system input, and $y(t)$ in R_p is the system output. The $A; B; C;$ and D are appropriately dimensioned real constant matrices. A dynamical system with single input ($m = 1$) and single output ($p = 1$) is called a SISO (single input and single output) system, otherwise it is called MIMO system.

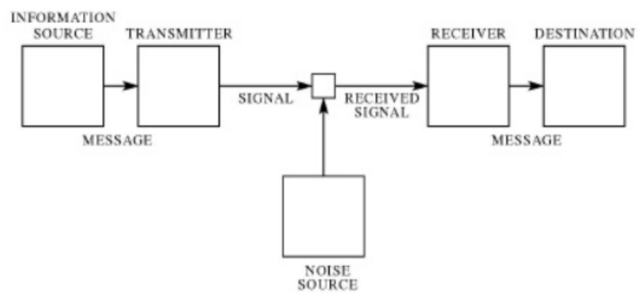

Figure 1: Blocks in a communication system. Credits: [3].

The following definitions will be useful in subsequent sections and parts.

Definition 1. The dynamical system described by Equation (1) or the pair $(A; B)$ is said to be controllable if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists a (piecewise continuous) input $u(\cdot)$ such that the solution of Equation (1) satisfies $x(t_1) = x_1$. Otherwise, the system or the pair $(A; B)$ is said to be uncontrollable.

Definition 2. An unforced dynamical system $\dot{x} = Ax$ is said to be stable if all the eigenvalues of A are in the open left half plane, i.e., $Re(A) < 0$. A matrix A with such a property is said to be stable or Hurwitz.

Definition 3. The dynamical system (Equation (1)), or the pair $(A; B)$, is said to be stabilizable if there exists a state feedback $u = Fx$ such that the system is stable, i.e., $A + BF$ is stable.

Definition 4. The dynamical system described by Equations (1) through (3) or by the pair $(C; A)$ is said to be observable if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the temporal history of the input $u(t)$ and the output $y(t)$ in the interval of $[0; t_1]$. Otherwise, the system, or $(C; A)$, is said to be unobservable.

2 Shannon's Formulation of Communication

Shannon presents a block diagram (Figure 1) summarizing a communication system [3]. The system consists of the message source, the encoder, a channel, the decoder and the destination. This is a point-to-point channel. In this formulation, how much delay is incurred in going from source to destination is immaterial; large end-to-end delay is permitted. For example, if the channel is specified as

$$y(t) = x(t) + n(t), \tag{4}$$

representing the operation occurring at the addition node, the overall delay in traveling to the addition node and then coming out of it is not mentioned/relevant.

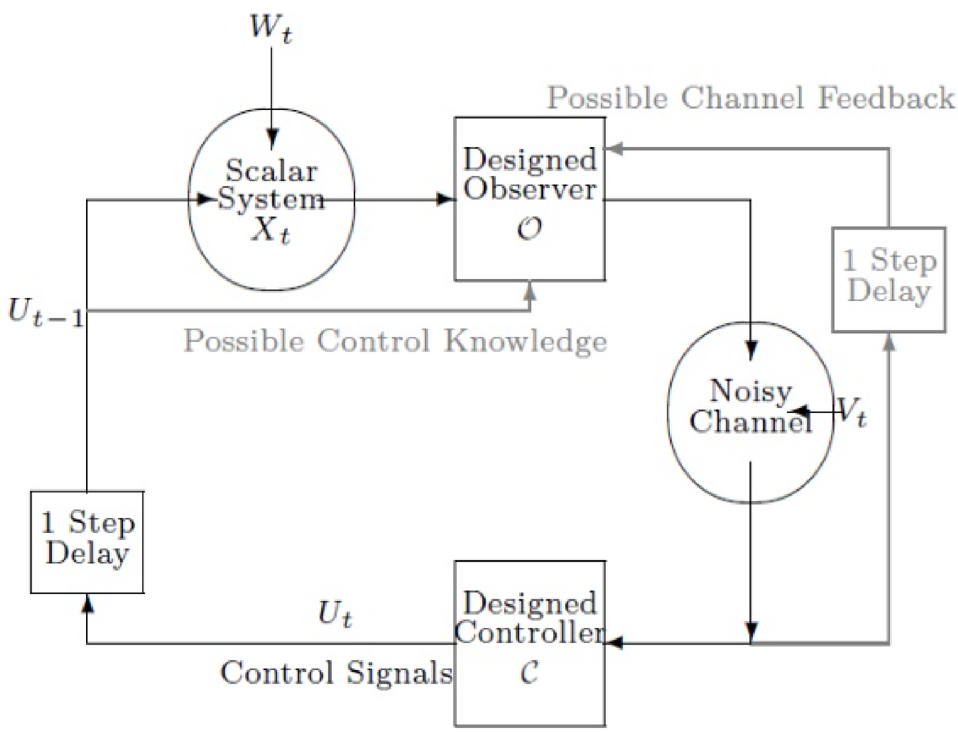

Figure 2: Distributed control setting. Credits: [4, 5].

3 A Scalar Distributed Control Problem

Consider a distributed control problem:

$$X_{t+1} = \lambda X_t + U_t + W_t \tag{5}$$

Such a problem arises when there are multiple controllers. So Sahai's diagram above might not depict a distributed system. Mere observer and controller separation might not make it a distributed control problem[1]. However, it is a scalar problem since no vectors are involved. In a separate paper [5] the same authors consider the vector version. Their goal is η-stability where the η-th power of the absolute value of the state must be bounded (finite)[4].

With reference to Equation (5), suppose if λ is greater than 1, the state would grow unboundedly, making it an unstable system. The stability requirement is that the η-th moment of the state be bounded from above for all time.

[1]Credits: a language model.

4 Is Shannon Capacity all we Need?

Consider a system with $\lambda = 2$ for the dynamics. $\lambda = 2$ means that the system state is doubling at every time step. What is a real erasure channel? Real refers to the fact that it carries real-valued numbers. Erasure refers to the fact that with some probability a transmission can be entirely dropped or erased, and replaced by some symbol 'E' called an erasure symbol. Thus,

$Z_t = Y_t$ with probability $\frac{1}{2}$,

$Z_t = 0$ with probability $\frac{1}{2}$.

Z_t is the output of the channel and Y_t is the input of the channel. Here the erasure symbol is '0.' There are no other constraints. The design is obvious: $Y_t = X_t$, $U_t = -\lambda Z_t$. We feed the state X_t into the channel input Y_t and feed the channel output Y_t, properly scaled, as the control signal U_t. The resulting closed-loop dynamics: $X_{t+1} = W_t$ with probability $\frac{1}{2}$, and $X_{t+1} = 2X_t + W_t$ with probability $1/2$. The closed loop dynamics are derived simply by appropriate substitutions.

5 Is the Closed-loop System Stable?

$X_{t+1} = W_t$ with probability $\frac{1}{2}$, and $X_{t+1} = 2X_t + W_t$ with probability $\frac{1}{2}$. i.i.d. erasures mean arbitrarily long stretches of erasures are possible, though unlikely. Since the erasures are i.i.d. with some probability, there can be a continuous string of erasures with some probability. Thus the lower option comes into play and the system is not guaranteed to stay inside any box.

Under stochastic disturbances, the variance of the state is asymptotically infinite. That is, system state can grow unboundedly and so the variance of the state is asymptotically infinite. Here we have employed the formula $Var(aX) = a^2 Var(X)$.

For worst case disturbances $W_t = 1$, the tail probability is dying off as $Pr(|X| > x) \equiv K/x$ where we have used Markov's inequality $Pr(|X| > x) \leq E[X]/x$. If we assume there is an equality instead of inequality, we get the bound on the tail as $E[X]/x$.

As one goes farther towards $\pm\infty$, the probability of being in the tail goes to 0. Thus the probability of large values is small. Nevertheless, the system is unboundedly and therefore uncontrollably growing, whereas the Shannon capacity $C = \infty$! The Shannon capacity is infinite, which means one can send any number of bits per second with 0 probability of error; thus Shannon capacity is *inadequate* as a notion useful when controlling a system by sending bits.

References

[1] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory.* 2005.

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[4] Anant Sahai and Sanjoy Mitter. The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link - Part I: Scalar systems. *IEEE Transactions on Information Theory*, 52(8):3369–3395, 2006.

[5] Anant Sahai and Sanjoy Mitter. The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link. part ii: vector systems,". *IEEE Trans. Inform. Theory*, 2005.