Capacity of a Neural PDE

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Abstract

In this review-note we are interested in determining the information bearing properties and abilities of the Hodgkin Huxley equation.

Introduction and Preliminaries

Ikeda and Manton [1] provide the capacity formula for a single spiking neuron channel. Here we are interested in a tract of N axons [2]. We start with the $N = 1$ case:

$$C\frac{\partial V}{\partial t} = G^{ax}\cos^2\theta\frac{\partial^2 V}{\partial z^2} - G^{my}V + I^{inj,ion} \tag{1}$$

and write its capacity as:

$$Cap = \max I(V(0,[0,T]);V(L,[0,T])). \tag{2}$$

Using $\phi_i(0,T)$ as a C.O.N. set of basis functions [3] we write $V(0,[0,T])$ as

$$\sum_{i=1}^{\infty} v_{0i}\phi_i(0,T). \tag{3}$$

Therefore $V(0,[0,T]) \equiv \{v_{0i}\}_{i=1}^{\infty}$, an equivalence in representation. Similarly, $V(L,(0,T)) \equiv \{v_{Li}\}_{i=1}^{\infty}$.Therefore, if

$$\alpha_i = f_{v_{0i}(v_{0i})}^{max} subj.toP \tag{4}$$

where P is the power constraint, then

$$Cap_i = \alpha_i I(\{v_{0i}\}_{i=1}^{\infty};\{v_{Li}\}_{i=1}^{\infty}). \tag{5}$$

Now, the condition for complete orthonormality is

$\int_{-\infty}^{\infty} \quad \phi_i(0,T) \quad \cdot\phi_j(0,T)dt = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

and

$$V(0,[0,T]) = \sum_{i=1}^{\infty} v_{0i}\phi_i(0,T). \tag{6}$$

Project both sides of Equation (6) onto the CON set, to get

$$\int_{-\infty}^{\infty} \phi_j(0,T)V(0,[0,T]) = \int_{-\infty}^{\infty} \phi_j(0,T)\sum_{i=1}^{\infty} v_{0i}\phi_i(0,T)$$

Switching the order of the summation and integration, we have

$$\begin{aligned}
&= \sum_{i=1}^{\infty} v_{0i}\int_{-\infty}^{\infty}\phi_i\phi_j dt\\
&= \sum_{i=1}^{\infty} v_{0i} \begin{matrix} 0, i\neq j \\ 1, i=j \end{matrix}\\
&= v_{0j}
\end{aligned}$$

Therefore,

$$v_{0j} = \int_{-\infty}^{\infty} \phi_j(0,T)\cdot V(0,[0,T])dt \tag{7}$$

and

$$v_{Lj} =< \phi_j(0,T), V(L,[0,T]) > . \tag{8}$$

Determining $H(v_{0j})$

Assume v_{0j} belong to a discrete set. This is possible if there are only a few stimulus waveforms. In this scenario, let $v_{0j} \in [v_{0j}^1, v_{0j}^2, \ldots, v_{0j}^k]$. Then,

$$H(v_{0j}) = -\sum_{i=1}^{k} p_i \log p_i \tag{9}$$

where $p_i = Pr(v_{0j} = v_{0j}^i)$. Assuming a uniform prior, we have $p_i = \frac{1}{k}$. Call this, assumption (**). This implies,

$$H(v_{0j}) = -\sum_{i=1}^{k} \frac{1}{k}\log(\frac{1}{k}), \tag{10}$$

where k is the total number of possible stimuli.

Determining $H(v_{0j}|v_{Lj})$

By definition,

$$H(v_{0j}|v_{Lj}) = -\mathbb{E}_{\mathrm{v_{Lj}}}[\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{Pr(v_{0j}^i|v_{Lj})\cdot \log(Pr(v_{0j}^i|v_{Lj}))}] \tag{11}$$

Determining $Pr(v_{0j}^i|v_{Lj})$

As per Bayes' formula [4],

$$Pr(v_{0j}^i|v_{Lj}) = \frac{Pr(v_{Lj}|v_{0j}^i) \cdot Pr(v_{0j}^i)}{Pr(v_{Lj})}. \tag{12}$$

Using the uniform prior, this yields

$$Pr(v_{0j}^i|v_{Lj}) = \frac{Pr(v_{Lj}|v_{0j}^i)}{k \cdot Pr(v_{Lj})}. \tag{13}$$

Further suppose that v_{Lj} also belong to a discrete set of size F. Then, $Pr(v_{Lj}|v_{0j}^i)$ which stands for $Pr(v_{Lj} = v_{Lj}^m|v_{0j}^i)$, can be equated with $\beta_{L|0}(m|i)$ which is the transition matrix of the 'channel.'

Next, we use the total probability theorem [5] to write,

$$Pr(v_{Lj} = v_{Lj}^m) = \sum_{i=1}^{k} \beta_{L|0}(m|i) \cdot Pr(i) \tag{14}$$

Now,

$$Pr(v_{0j}^i|v_{Lj}^m) = \frac{Pr(v_{Lj}|v_{0j}^i)}{k \cdot Pr(v_{Lj}^m)} = \frac{\beta_{L|0}(m|i)}{\sum_{n=1}^{k} \beta_{L|0}(m|n)} \tag{15}$$

This allows us to determine $H(v_{0j}|v_{Lj})$. Thus, we in essence have $I(v_{0i}; v_{Li})$.

Next Steps and Conclusion

Our next step is to relax assumption (**), in order to get the general mutual information. Then we may maximize it over choices of $\{p_i\}$ subject to power or cost constraints. Numerical simulations can also be invoked and might be useful, especially for $N > 1$.

References

[1] Shiro Ikeda and Jonathan H Manton. Capacity of a single spiking neuron channel. *Neural Computation*, 21(6):1714–1748, 2009.

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[5] Athanasios Papoulis and S Unnikrishna Pillai. *Probability, random variables and stochastic processes.* 2002.