Behavior Near Capacity: A Study of Error Exponents

Aman Chawla

March 6, 2024

Abstract

In this note the authors study the higher order exponential behavior of the probability of error decay with blocklength and especially the behavior at rates approaching capacity. It is found that exponents of high order also are very informative. In particular, the second order exponent for the Gaussian channel shows a panicked rise near capacity indicating the rush to decelerate the first order exponent to zero.

In studying error exponents of various channels, such as classical channels, quantum channels, classical-quantum channels and so on, one comes across a number of interesting types of exponents. For example, there is the usual exponent, but there is also the one that decays as a double exponential. In this paper we asked the question whether such "double" or even "triple" exponential behavior could be found in normal error exponents. To answer the question we formulate the general error exponent of order N as follows:

$$E_N(R) = -\frac{\log(\log(\ldots\log(Poe)))}{B} \tag{1}$$

where Poe is the probability of error and B is the blocklength or decoding time. The logarithms are nested N times in total. We apply this to a known system, such as the Gaussian channel. Given that it has a normal exponent rather than a doubly exponential exponent, we don't expect anything much out of this application, but we do find it. In particular, as Figure 1 shows, we see that for $N = 2$, we get interesting behavior even for the standard reliability function of the AWGN channel. Near capacity, the $N = 2$ exponent blows up. If we think of the $N = 2$ exponent as the 'deceleration' while the $N = 1$ exponent is just the velocity, then we get some intuition. The system is applying the brakes to bring down the 'velocity' to zero at capacity. In future work we intend to study these higher order exponents for various channels, classical and quantum and develop a better sense of their meaning in a practical setting. This may also have relevance for the finite blocklength setting [1].

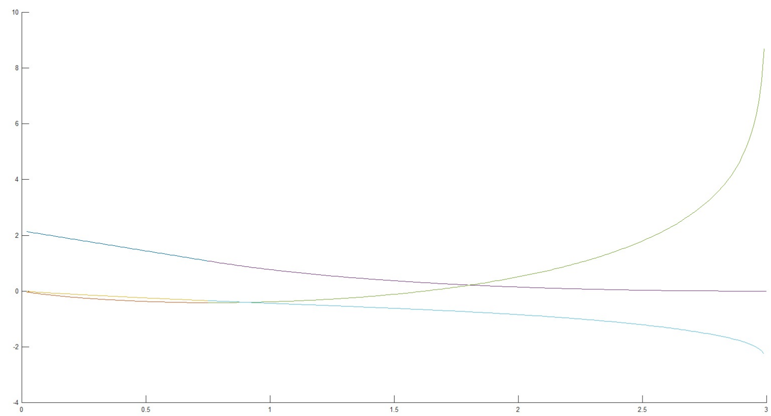

Figure 1: Higher order exponent curves for the standard gaussian noise channel. The vertical axis is the error exponent and the horizontal axis is the rate in bits per second. Seen near the capacity value of 3 bits per second, the topmost curve is the $N = 2$ case, the middle one is the $N = 1$ case and the bottom one is the $N = 3$ case.

References

[1] Yury Polyanskiy, H Vincent Poor, and Sergio Verdú. Channel coding rate in the finite blocklength regime. *IEEE Transactions on Information Theory*, 56(5):2307–2359, 2010.