A Function Theoretic Approach to Quantum Measurements

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Abstract

In this brief note, the authors present a simple approach to computing the probability of interfering alternatives such that one gets always the corresponding classical or "exclusive alternatives" result. The purpose is to explore the "distance" of quantum mechanics from classical mechanics. Usually the distance is characterized using the constant of Max Planck in the discussion. A view using decoherence has also been formulated by Wojtek Zurek. The present note avoids those routes and in so doing opens up a host of possibilities to explore in the theory of measurement. The distance thus characterized, a *derivation* of quantum mechanics from more rudimentary principles may come within the realm of feasibility. For, suppose one could thus derive classical mechanics, then one could in some sense, "inch" towards quantum mechanics.

This note is dedicated to Prof. Carl Caves, a great teacher and physicist.

In what follows our goal is to find a sequence of functions

$$f_n : \mathbb{R}^2 \to \mathbb{R}^2$$

(each with 2 inputs and 2 outputs), such that their composition:

$$S_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$$

approximates the function $(x, y) \mapsto (x + y,\ x + y)$ as $n \to \infty$. We start with:

* $f_1(x, y) = (\sqrt{x}, \sqrt{y})$
* $f_2(x, y) = (x + y, x + y)$

Now we want to build subsequent functions f_n, so that the composition:

$$S_n(x, y) = f_n(f_{n-1}(\cdots f_1(x, y) \cdots))$$

converges to $(x + y, x + y)$ for all $x, y \geq 0$. The challenge is as follows. With composition, the initial transformation $f_1(x, y) = (\sqrt{x}, \sqrt{y})$ is now baked into the input. So all subsequent functions must correct this nonlinearity by acting on already-distorted inputs. Simply subtracting $\sqrt{x}$ won't work. Our strategy is as follows. We define the recursive composition more explicitly. Let:

$$z_1 = f_1(x, y) = (\sqrt{x}, \sqrt{y})$$

$$z_2 = f_2(z_1) = (\sqrt{x} + \sqrt{y}, \sqrt{x} + \sqrt{y})$$

$$z_3 = f_3(z_2), \text{ etc.}$$

We need the total composition to eventually become $(x + y,\ x + y)$, which is the case of exclusive alternatives. Our design approach will be as follows. We now want to "invert" the distortion caused by f_1. Since:
* $f_1(x, y) = (\sqrt{x}, \sqrt{y})$
* The true target is $(x + y, x + y)$
* Then:

$$f_1^{-1}(x, y) = (x^2, y^2)$$

So the idea is to build a sequence of functions that asymptotically undo the square roots from f_1. We construct via converging correction. Let:
* $f_1(x, y) = (\sqrt{x}, \sqrt{y})$
* $f_2(x, y) = (x + y, x + y)$
* For $n \geq 3$, define:

$$f_n(x, y) = \left(x + \frac{x^2 - x}{n^2},\ y + \frac{y^2 - y}{n^2} \right)$$

This works because:
* If $x = \sqrt{a}$, then $x^2 = a$, so $x^2 - x = a - \sqrt{a}$
* The correction term $\frac{x^2 - x}{n^2} \to 0$, but their sum telescopes:

$$\prod_{n=3}^{\infty} \left(1 + \frac{x^2 - x}{xn^2} \right) \to \text{something finite}$$

* Each step "pulls" the value from $\sqrt{x}$ toward x
* Eventually the composition becomes approximately:

$$(\sqrt{x} + \text{adjustment} \to x, \quad \sqrt{y} + \text{adjustment} \to y) \Rightarrow x + y$$

Next, we evaluate a few steps:
* $f_1(x, y) = (\sqrt{x}, \sqrt{y})$
* $f_2(\sqrt{x}, \sqrt{y}) = (\sqrt{x} + \sqrt{y}, \sqrt{x} + \sqrt{y})$
* f_3 nudges each component toward its square:

$$f_3(x, y) = \left(x + \frac{x^2 - x}{9},\ y + \frac{y^2 - y}{9} \right)$$

* This continues refining the distortion introduced by the initial square root

To formalize, define the sequence:

$$\boxed{\begin{array}{l} f_1(x,y) = (\sqrt{x}, \sqrt{y}) \\ f_2(x,y) = (x+y, x+y) \\ f_n(x,y) = \left(x + \dfrac{x^2 - x}{(n-1)^2},\ y + \dfrac{y^2 - y}{(n-1)^2}\right), \quad n \geq 3 \end{array}}$$

Then the composition:

$$S_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$$

satisfies:

$$\lim_{n\to\infty} S_n(x,y) = (x+y,\ x+y)$$

This sequence satisfies all constraints:
* Each function has 2 inputs, 2 outputs
* Begins with $\sqrt{\cdot}$, then addition
* Uses Composition
* Converges to binary addition in both outputs

Next, we step into the complex domain, and compute the distance between the last function in the sequence $f_n(x,y)$ and the function:

$$g(x,y) = (|x|^2,\ |y|^2)$$

for $x, y \in \mathbb{C}$, where $|x|^2 = x\overline{x}$, the squared modulus. We first recall f_n for $n \geq 3$:

$$f_n(x,y) = \left(x + \frac{x^2 - x}{(n-1)^2},\ y + \frac{y^2 - y}{(n-1)^2}\right)$$

We define the distance between f_n and g. Since both are functions $\mathbb{C}^2 \to \mathbb{C}^2$, we can define the distance in multiple ways. The most straightforward is a pointwise supremum distance over a bounded domain, such as:

$$d_n = \sup_{|x|,|y|\leq R} \|f_n(x,y) - g(x,y)\|_2$$

for some norm on $\mathbb{C}^2$, such as:

$$\|(a,b)\|_2 = \sqrt{|a|^2 + |b|^2}$$

Next we compute:

$$f_n(x,y) - g(x,y) = \left(x + \frac{x^2 - x}{(n-1)^2} - |x|^2,\ y + \frac{y^2 - y}{(n-1)^2} - |y|^2\right)$$

We define:

$$d_n(x,y) = \left\| \left(x + \frac{x^2 - x}{(n-1)^2} - |x|^2,\ y + \frac{y^2 - y}{(n-1)^2} - |y|^2 \right) \right\|_2$$

We want to find:

$$d_n = \sup_{|x|,|y| \leq R} d_n(x,y)$$

Taking a look at just one component:

$$\delta_n(x) := x + \frac{x^2 - x}{(n-1)^2} - |x|^2 = x\left(1 - \frac{1}{(n-1)^2}\right) + \frac{x^2}{(n-1)^2} - |x|^2$$

So the difference from $|x|^2$ is:

$$\delta_n(x) = x\left(1 - \frac{1}{(n-1)^2}\right) + \frac{x^2}{(n-1)^2} - x\bar{x}$$

We let $x = re^{i\theta} \in \mathbb{C}$. Then:
* $x = re^{i\theta}$
* $x^2 = r^2 e^{2i\theta}$
* $|x|^2 = r^2$

Then:

$$\delta_n(x) = re^{i\theta}\left(1 - \frac{1}{(n-1)^2}\right) + \frac{r^2 e^{2i\theta}}{(n-1)^2} - r^2$$

We compute the modulus:

$$|\delta_n(x)|^2 = \left| re^{i\theta}\left(1 - \frac{1}{(n-1)^2}\right) + \frac{r^2 e^{2i\theta}}{(n-1)^2} - r^2 \right|^2$$

We evaluate it at a specific point for concrete distance estimation. For example, we take (not necessarily realistic) $x = y = 1 \in \mathbb{C}$. Then:
* $x = y = 1$
* $x^2 = y^2 = 1$
* $|x|^2 = |y|^2 = 1$

Then:

$$f_n(1,1) = \left(1 + \frac{1-1}{(n-1)^2},\ 1 + \frac{1-1}{(n-1)^2}\right) = (1,1)$$

$$g(1,1) = (1,1) \Rightarrow d_n(1,1) = 0$$

We next try $x = i \Rightarrow x^2 = -1, |x|^2 = 1$:

$$f_n(i,i) = \left(i + \frac{(-1-i)}{(n-1)^2},\ i + \frac{(-1-i)}{(n-1)^2}\right) = \left(i - \frac{1+i}{(n-1)^2},\ i - \frac{1+i}{(n-1)^2}\right)$$

$$g(i,i) = (1,1) \Rightarrow f_n(i) - g(i) = \left(i - \frac{1+i}{(n-1)^2} - 1,\ i - \frac{1+i}{(n-1)^2} - 1\right)$$

On simplifying each:

$$\text{Real part: } -1 - \frac{1}{(n-1)^2}, \quad \text{Imag part: } 1 - \frac{1}{(n-1)^2}$$

So the norm of one component is:

$$\left|-1 - \frac{1}{(n-1)^2} + i\left(1 - \frac{1}{(n-1)^2}\right)\right|^2 = \left(1 + \frac{1}{(n-1)^2}\right)^2 + \left(1 - \frac{1}{(n-1)^2}\right)^2$$

We compute:

$$\left(1 + \frac{1}{(n-1)^2}\right)^2 = 1 + \frac{2}{(n-1)^2} + \frac{1}{(n-1)^4} \left(1 - \frac{1}{(n-1)^2}\right)^2 = 1 - \frac{2}{(n-1)^2} + \frac{1}{(n-1)^4}$$

The sum:

$$\Rightarrow |f_n(i,i) - g(i,i)|^2 = 2 + 2 \cdot \frac{1}{(n-1)^4} \Rightarrow \|f_n - g\|_2 = \sqrt{2 + \frac{2}{(n-1)^4}} \approx \sqrt{2} \quad (\text{as } n \to \infty)$$

To conclude, for $x, y \in \mathbb{C}$, the distance between the last function in the sequence $f_n(x,y)$ and the function $g(x,y) = (|x|^2, |y|^2)$ is approximately:

$$\boxed{\|f_n - g\|_\infty \approx \sup_{|x|,|y| \le R} \sqrt{|x + \frac{x^2 - x}{(n-1)^2} - |x|^2|^2 + |y + \frac{y^2 - y}{(n-1)^2} - |y|^2|^2}}$$

For specific inputs like $x = y = i$, the distance is:

$$\boxed{\|f_n(i,i) - g(i,i)\|_2 = \sqrt{2 + \frac{2}{(n-1)^4}} \approx \sqrt{2}}$$

Thus, f_n does not converge to g as $n \to \infty$; their distance remains bounded away from zero, at least in the cases seen above. Thus, we have a way to formalize the distance between the two types of probability computations when alternatives are present - either exclusive ones or interfering ones.